ON THE REDUCTIONS OF CERTAIN TWO-DIMENSIONAL CRYSTALLINE REPRESENTATIONS, II

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Abstract. In a recent paper [LTXZ], Liu–Truong–Xiao–Zhao proved several conjectures (including the Breuil–Buzzard–Emerton conjecture and Gouvêa's conjecture on slopes of modular forms) by computing the slopes $v_p(a)$ of certain local representations $\overline{V}_{k,a}$. In this article, we complement the results in [LTXZ] by explicitly computing the representations $\overline{V}_{k,a}$ themselves, for many small slopes $v_p(a)$.

1. Introduction and results

Let p be an odd prime number and $k \geqslant 2$ be an integer, and let a be an element of $\overline{\mathbb{Z}}_p$ such that $v_p(a) > 0$. There has been considerable interest in computing certain local representations $\overline{V}_{k,a}$ which arise as representations associated with modular forms (see [LTXZ], [Ars21], [Ber10], [Bre03a], [Bre03b], [Edi92], [BLZ04], [BG15], [BG09], [BG13], [GG15]). In particular, a conjecture of Breuil–Buzzard–Emerton dating back to 2005 states that, if k is even and the slope $v_p(a)$ is not an integer, then $\overline{V}_{k,a}$ is irreducible. This conjecture was finally proved by Liu–Truong–Xiao–Zhao in [LTXZ]. The goal of this article is to further explore the Breuil–Buzzard–Emerton conjecture and complement the result in [LTXZ] by explicitly computing many examples of $\overline{V}_{k,a}$, for many small slopes $v_p(a)$.

1.1. Background.

The main objects we study are irreducible two-dimensional crystalline representations of the absolute Galois group of $\mathbb{Q}_p$. These are up to a twist parametrized by an integer $k \geqslant 2$ and an element $a \in \overline{\mathbb{Z}}_p$ such that $v_p(a) > 0$, and we denote by $V_{k,a}$ the representation corresponding to the parameters (k, a). An explicit construction of $V_{k,a}$ is given in Subsection 3.1 of [Bre03b]. Therefore, all two-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ are of the form $V_{k,a} \otimes \eta$ for some character η of $G_{\mathbb{Q}_p}$ that is the product of an unramified character and a power of the cyclotomic character. We define $\overline{V}_{k,a}$ as the semi-simplification of the reduction modulo the maximal ideal $\mathfrak{m}$ of $\overline{\mathbb{Z}}_p$ of a Galois stable $\overline{\mathbb{Z}}_p$-lattice in $V_{k,a}$ (with the resulting representation being independent of the choice of lattice). The question of computing $\overline{V}_{k,a}$ has been studied extensively, and we refer to the introduction of [Ars21] for a brief exposition of it.

In the article [Ars21], these representations have been computed over certain "non-subtle" components of weight space. We say that a weight k belongs to a "non-subtle" component of weight space if and only if

$$k \not\equiv 3, 4, \ldots, 2\nu, 2\nu + 1 \bmod p - 1.$$

Thus there are $\max\{\frac{p-1}{2} - \nu + 1, 0\}$ many "non-subtle" components of weight space. This article is a continuation of [Ars21], in which we completely classify these representations over the "non-subtle" components of weight space, both for integer and non-integer slopes. Before stating the main results, let us first introduce some terminology.

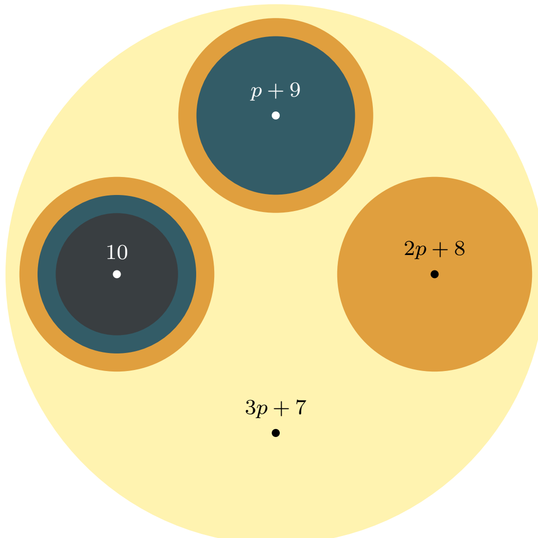
Let us denote $\nu = \lfloor v_p(a) \rfloor + 1 \in \mathbb{Z}_{>0}$. We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$ and to look at the space of continuous homomorphisms $\mathscr{W} = \hom_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. We have $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$, and $\hom_{cts}(\mathbb{Z}_p, \mathbb{C}_p^\times)$ is isomorphic to the open disk in $\mathbb{C}_p$ with center 0 and radius 1, via the identification $\chi \leftrightarrow \chi(1) - 1$. Thus $\mathscr{W}$ is the disjoint union of $p-1$ open disks of radii 1. We identify the weight $k \geqslant 2$ with the continuous homomorphism $x \mapsto x^{k-2}$, so that $k \geqslant 2$ is a point on the disk indexed by $k - 2 \bmod p - 1$. We say a point of $\mathscr{W}$ is "integral" if it is associated with a weight in this fashion.

The main theorem in [BLZ04] computes $\overline{V}_{k,a}$ in the neighborhood of $a = 0$:

$$\overline{V}_{k,a} \cong \overline{V}_{k,0} \cong \begin{cases} \operatorname{ind}(\omega_2^{k-1}) & \text{if } p+1 \nmid k-1, \\ (\mu_{\sqrt{-1}} \oplus \mu_{-\sqrt{-1}}) \otimes \omega^{\frac{k-1}{p+1}} & \text{if } p+1 \mid k-1, \end{cases}$$

whenever $v_p(a) > \lfloor \frac{k-2}{p-1} \rfloor$. This theorem tells us what $\overline{V}_{k,a}$ is at a discrete set of points in the aforementioned $p-1$ disks.

The theorems we prove indicate that these points play a fundamental role: it seems that the $p-1$ open disks can be split into regions by concentric circles centered at these points in a way that $\overline{V}_{k,a}$ depends on the region k belongs to. For example, the following diagram illustrates how the disk containing the weight 10 is split into regions when $p \geqslant 11$ and $3 < v_p(a) < 4$. There are four centers of the bundles of concentric circles, and they are exactly the points k belonging to the disk (i.e. such that $k \equiv 10 \bmod p-1$) and such that $v_p(a) > \lfloor \frac{k-2}{p-1} \rfloor$ (i.e. $\lfloor \frac{k-2}{p-1} \rfloor < 4$). In fact, they are the same as the points satisfying $\frac{k-1}{p+1} < 4$. By [BLZ04], at these four points we have $\overline{V}_{k,a} \cong \operatorname{ind}(\omega_2^{k-1})$. The diagram illustrates closed disks around the points, of radii p^{-1}, p^{-2}, and p^{-3}. As we show in theorem 1, in this situation $\overline{V}_{k,a}$ is always one of these four representations, depending on the color of the region that k belongs to, as illustrated.



Theorems 1 and 2 prove a general version of this, in the case when the corresponding disk is what we label "non-subtle": this label depends on the slope $v_p(a)$ and roughly means that the bound $\lfloor \frac{k-2}{p-1} \rfloor$ is optimal for any k belonging to that disk, i.e. that there are no additional points k on that disk satisfying the improved bound $v_p(a) > \frac{k-1}{p+1}$ but not $v_p(a) > \lfloor \frac{k-2}{p-1} \rfloor$ at which the associated representation is distinct from the "typical" one, which in the context of the diagram on the previous page means the representation corresponding to the yellow region.

Let us write $\overline{h}$ for the number in $\{1, \ldots, p-1\}$ which is congruent to h mod $p-1$. Let $\nu = \lfloor v_p(a) \rfloor + 1 \in \mathbb{Z}_{>0}$, and let s be the number in $\{1, \ldots, p-1\}$ which is congruent to $k-2$ mod $p-1$. Let us say that k is "subtle" if $s \in \{1, \ldots, 2\nu-1\}$, and k is "non-subtle" if $s \notin \{1, \ldots, 2\nu-1\}$. An open disk of $\mathscr{W}$ consists either entirely of "subtle" points or entirely of "non-subtle" points, so we can also refer to the $p-1$ open disks of $\mathscr{W}$ as either "subtle" or "non-subtle". In particular, the $\min\{p-1, 2\nu-1\}$ disks containing $3, \ldots, 2\nu+1$ are "subtle", and all other disks are "non-subtle". Note that whether a weight is "subtle" or not depends on the value of ν.

Let $\mathscr{D}_s$ denote the open disk of radius 1 around $s+2 \in \mathscr{W}$, and let us consider the set

$$B_{s,\nu} = \{s + \beta(p-1) + 2 \mid \beta \in \{0, \ldots, \nu-2\}\}.$$

In particular, if $\nu = 1$ then $B_{s,\nu} = \varnothing$, and in general $B_{s,\nu}$ is a set of $\nu - 1$ points in $\mathscr{D}_s$. Let

$$b_0 < \cdots < b_{\nu-2}$$

be the elements of $B_{s,\nu}$ in increasing order. Therefore if $i \in \{0, \ldots, \nu-2\}$ then $b_i = s + i(p-1) + 2$ and, by the main result of [BLZ04],

$$\overline{V}_{b_i,a} \cong \mathrm{ind}(\omega_2^{b_i - 1}).$$

Let us also define $b_{\nu-1} = s + (\nu-1)(p-1) + 2$. For $i \in \{0, \ldots, \nu-2\}$ and $j \in \mathbb{Z}_{>0}$, let

$$\mathscr{R}_{i,j}^{s,\nu} = \{t \in \mathscr{D}_s \mid j \leqslant v_p(t - b_i) < j+1\}$$

be the half-open annulus which is the complement of the closed disk of radius p^{-j-1} around b_i in the closed disk of radius p^{-j} around b_i. The integral points in $\mathscr{R}_{i,j}^{s,\nu}$ are the points on the circle of radius p^{-j} around $b_i = s + i(p-1) + 2$. Finally, let

$$\mathscr{R}_0^{s,\nu} = \mathscr{D}_s \setminus \cup_{i\in\{0,\dots,\nu-2\},\, j>0} \mathscr{R}_{i,j}^{s,\nu},$$

so that $\mathscr{D}_s$ is partitioned into the disjoint sets

$$(\mathscr{R}^{s,\nu}) \qquad \{\mathscr{R}_0^{s,\nu}\} \cup \{\mathscr{R}_{i,j}^{s,\nu} \mid i \in \{0,\dots,\nu-2\},\, j \in \mathbb{Z}_{>0}\}.$$

Note that the definition of this partition depends on both s and ν. For $l \in \mathbb{Z}$ and $\lambda \in \overline{\mathbb{F}}_p^\times$ let us define

$$\mathrm{Irr}(l) = \mathrm{ind}(\omega_2^{l-1}) \text{ and } \mathrm{Red}_{s,\nu}(l,\lambda) = \mu_\lambda \omega^{s+l-\nu+2} \oplus \mu_{\lambda^{-1}} \omega^{\nu-l-1}.$$

The first result is a complete classification of $\overline{V}_{k,a}$ over the "non-subtle" components of weight space for $v_p(a) \notin \mathbb{Z}$.

Theorem 1. *Recall that $k \geqslant 2$ is an integer and that s is defined as the integer in $\{1,\dots,p-1\}$ which is congruent to $k-2 \bmod p-1$. Suppose that k is "non-subtle", i.e.*

$$k \not\equiv 3, 4, \dots, 2\nu, 2\nu+1 \bmod p-1.$$

Suppose also that the open disk $\mathscr{D}_s$ of radius 1 around $s+2 \in \mathscr{W}$ is partitioned into disjoint sets as in $(\mathscr{R}^{s,\nu})$. If $v_p(a) \notin \mathbb{Z}$ then

$$\overline{V}_{k,a} \cong \begin{cases} \mathrm{Irr}(b_{\nu-1}) & \text{if } k \in \mathscr{R}_0^{s,\nu}, \\ \mathrm{Irr}(b_{\max\{i,\nu-j-1\}}) & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu}. \end{cases}$$

This result is known for $\nu = 1$ by the work of Buzzard and Gee in [BG09] and for $\nu = 2$ by the work of Bhattacharya and Ghate in [BG15].

We also prove a similar theorem for $v_p(a) \in \mathbb{Z}$. The precise statement of this is the following theorem, which is a complete classification of $\overline{V}_{k,a}$ over the "non-subtle" components of weight space for $v_p(a) \in \mathbb{Z}$.

Theorem 2. *Recall that $k \geqslant 2$ is an integer and that s is defined as the integer in $\{1,\dots,p-1\}$ which is congruent to $k-2 \bmod p-1$. Suppose that k is "non-subtle", i.e.*

$$k \not\equiv 3, 4, \dots, 2\nu, 2\nu+1 \bmod p-1.$$

Suppose also that the open disk $\mathscr{D}_s$ of radius 1 around $s+2 \in \mathscr{W}$ is partitioned into disjoint sets as in $(\mathscr{R}^{s,\nu})$. If $v_p(a) = \nu - 1 \in \mathbb{Z}_{>0}$ then

$$\overline{V}_{k,a} \cong \begin{cases} \mathrm{Red}_{s,\nu}(0,\lambda_{k,\nu}) & \text{if } k \in \mathscr{R}_0^{s,\nu}, \\ \mathrm{Red}_{s,\nu}(j,\lambda_{k,\nu,i,j}) & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu} \text{ and } i+j < \nu-1, \\ \mathrm{Irr}(b_i) & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu} \text{ and } i+j \geqslant \nu-1, \end{cases}$$

where

$$\lambda_{k,\nu} = \frac{\binom{s-\nu+2}{\nu-1} a}{\binom{s-k+2}{\nu-1} p^{\nu-1}} \in \overline{\mathbb{F}}_p^\times,$$

$$\lambda_{k,\nu,i,j} = \frac{(-1)^{\nu+i+j+1}(\nu-j-1)\binom{\nu-j-2}{i}\binom{s-\nu+j+2}{\nu-j-1} a}{(k-s-i(p-1)-2)p^{\nu-j-1}} \in \overline{\mathbb{F}}_p^\times.$$

Note that $\lambda_{k,\nu,i,j}$ is indeed a unit (and therefore we can think of it as an element of $\overline{\mathbb{F}}_p^\times$) since the integral points in $\mathscr{R}_{i,j}^{s,\nu}$ consist precisely of those points on the circle of radius p^{-j} around $b_i = s + i(p-1) + 2$ and so

$$v_p((k - s - i(p-1) - 2)p^{\nu-j-1}) = \nu - 1.$$

This result is known for $\nu = 2$ by the work of Bhattacharya, Ghate, and Rozensztajn in [BGR18].

The paper [Roz18] gives an algorithm which takes as input a prime p, a weight k, an eigenvalue a, and a parameter called "radius" which determines the precision of the computations, and if the radius is large enough it computes $\overline{V}_{k,a}$.[1] An implementation of the algorithm in `SageMath` is available at

`http://perso.ens-lyon.fr/sandra.rozensztajn/software/index.html`

As theorems 1 and 2 give complete classifications of $\overline{V}_{k,a}$, one can use this algorithm to verify them for any given triple (p,k,a). The complexity of the algorithm depends on the size of the extension field generated by a, so in practice it is much faster to verify theorem 2. Additionally, the statement of theorem 2 is more complicated, especially the formulas for $\lambda_{k,\nu}, \lambda_{k,\nu,i,j}$, so it is better suited for this type of computer verification. We have verified theorem 2 for the triples in the following table.

p	k	a	"radius"	$\overline{V}_{k,a}$
7	8	49	3	$\mathrm{ind}(\omega_2^7)$
7	14	49	3	$\mathrm{ind}(\omega_2^{13})$
7	20	49	4	$\mu_3\omega^5 \oplus \mu_5\omega^2$
7	26	49	3	$\mu_1\omega^5 \oplus \mu_1\omega^2$
7	32	49	3	$\mu_4\omega^5 \oplus \mu_2\omega^2$
7	38	49	3	$\mu_1\omega^5 \oplus \mu_1\omega^2$
7	44	49	3	$\mu_3\omega^5 \oplus \mu_5\omega^2$
7	50	49	3	$\mu_6\omega \oplus \mu_6$
7	56	49	3	$\mathrm{ind}(\omega_2^{13})$
11	38	121	3	$\mu_7\omega^5 \oplus \mu_8\omega^2$
11	39	121	3	$\mu_5\omega^6 \oplus \mu_9\omega^2$
11	40	121	3	$\mu_7\omega^7 \oplus \mu_8\omega^2$
11	41	121	3	$\mu_2\omega^8 \oplus \mu_6\omega^2$
11	42	121	3	$\mu_1\omega^9 \oplus \mu_1\omega^2$

[1]The algorithm actually computes the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated with $\overline{V}_{k,a}$ via the bijective correspondence given in Theorem 2 in [Ars21], and one can use Theorem 2 in [Ars21] to then compute $\overline{V}_{k,a}$.

2. Computing $\overline{V}_{k,a}$ by computing $\overline{\Theta}_{k,a}$

From now on we assume the notation from sections 2, 3, 4, and 6 of [Ars21]. Moreover, we assume that $k > p^{100}$ as in section 5 of [Ars21]. For $l \in \mathbb{Z}$ let us define

$$\mathrm{BIrr}(l) = \left(\mathrm{ind}_{KZ}^{G} \sigma_{l_1}/T\right) \otimes \omega^{l_2},$$

where l_1 and l_2 are the unique integers such that

$$l = l_1 + (p+1)l_2 + 2 \text{ and } l_1 \in \{0, \ldots, p-1\}.$$

For $l \in \mathbb{Z}$ and $\lambda \in \overline{\mathbb{F}}_p^{\times}$ let us define

$$\begin{aligned}&\mathrm{BRed}_{s,\nu}(l,\lambda)\\ &\quad = \pi(\underline{s+2l-2\nu+2}, \lambda, \omega^{\nu-l-1}) \oplus \pi(\underline{2\nu-s-2l-4}, \lambda^{-1}, \omega^{s+l-\nu+2}).\end{aligned}$$

Theorem 2 in [Ars21] implies that our main theorems can be rewritten in the following equivalent forms. Recall that we assume $p > 2$ throughout.

Theorem 3. *Recall that $k \geqslant 2$ is an integer and that s is defined as the integer in $\{1, \ldots, p-1\}$ which is congruent to $k-2 \bmod p-1$. Suppose that k is "non-subtle", i.e.*

$$k \not\equiv 3, 4, \ldots, 2\nu, 2\nu+1 \bmod p-1.$$

Suppose also that the open disk $\mathscr{D}_s$ of radius 1 around $s+2 \in \mathscr{W}$ is partitioned into disjoint sets as in $(\mathscr{R}^{s,\nu})$. If $v_p(a) \notin \mathbb{Z}$ then

$$\overline{\Theta}_{k,a} \cong \begin{cases} \mathrm{BIrr}(b_{\nu-1}) & \text{if } k \in \mathscr{R}_0^{s,\nu}, \\ \mathrm{BIrr}(b_{\max\{i,\nu-j-1\}}) & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu}. \end{cases}$$

Theorem 4. *Recall that $k \geqslant 2$ is an integer and that s is defined as the integer in $\{1, \ldots, p-1\}$ which is congruent to $k-2 \bmod p-1$. Suppose that k is "non-subtle", i.e.*

$$k \not\equiv 3, 4, \ldots, 2\nu, 2\nu+1 \bmod p-1.$$

Suppose also that the open disk $\mathscr{D}_s$ of radius 1 around $s+2 \in \mathscr{W}$ is partitioned into disjoint sets as in $(\mathscr{R}^{s,\nu})$. If $v_p(a) = \nu - 1 \in \mathbb{Z}_{>0}$ then

$$\overline{\Theta}_{k,a}^{\mathrm{ss}} \cong \begin{cases} \mathrm{BRed}_{s,\nu}(0, \lambda_{k,\nu})^{\mathrm{ss}} & \text{if } k \in \mathscr{R}_0^{s,\nu}, \\ \mathrm{BRed}_{s,\nu}(j, \lambda_{k,\nu,i,j})^{\mathrm{ss}} & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu} \text{ and } i+j < \nu-1, \\ \mathrm{BIrr}(b_i) & \text{if } k \in \mathscr{R}_{i,j}^{s,\nu} \text{ and } i+j \geqslant \nu-1, \end{cases}$$

where

$$\begin{aligned}\lambda_{k,\nu} &= \frac{\binom{s-\nu+2}{\nu-1} a}{\binom{s-k+2}{\nu-1} p^{\nu-1}} \in \overline{\mathbb{F}}_p^{\times}, \\ \lambda_{k,\nu,i,j} &= \frac{(-1)^{\nu+i+j+1}(\nu-j-1)\binom{\nu-j-2}{i}\binom{s-\nu+j+2}{\nu-j-1} a}{(k-s-i(p-1)-2)p^{\nu-j-1}} \in \overline{\mathbb{F}}_p^{\times}.\end{aligned}$$

Thus our task is to prove theorems 3 and 4.

3. Combinatorics

Throughout the proof we will refer to the combinatorial results in section 8 of [Ars21]. For convenience, we reproduce the statements here in the form we will use.

Lemma 5. *Suppose throughout this lemma that*

$$n, t, y \in \mathbb{Z}, \qquad b, d, k, l, w \in \mathbb{Z}_{\geqslant 0}, \qquad m, u, v \in \mathbb{Z}_{\geqslant 1}.$$

(1) *If $u \equiv v \bmod (p-1)p^{m-1}$ then*

$$\mathrm{M}_{u,n} \equiv \mathrm{M}_{v,n} \bmod p^m. \tag{c-a}$$

(2) *Suppose that $u = t_u(p-1) + s_u$ with $s_u = \overline{u}$, so that $s_u \in \{1, \dots, p-1\}$ and $t_u \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\mathrm{M}_u = 1 + [u \equiv_{p-1} 0] + \tfrac{t_u}{s_u} p + \mathsf{O}\big(t_u p^2\big). \tag{c-b}$$

(3) *If $n \leqslant 0$ then*

$$\mathrm{M}_{u,n} = \textstyle\sum_{i=0}^{-n} (-1)^i \binom{-n}{i} \mathrm{M}_{u-n-i,0}. \tag{c-c}$$

(4) *If $n \geqslant 0$ then*

$$\mathrm{M}_{u,n} \equiv (1 + [u \equiv_{p-1} n \equiv_{p-1} 0]) \binom{\overline{u}}{\underline{n}} \bmod p. \tag{c-d}$$

(5) *If $u \geqslant (b+l)d$ and $l \geqslant w$ then*

$$\textstyle\sum_j (-1)^{j-b} \binom{l}{j-b} \binom{u-dj}{w} = [w = l] d^l. \tag{c-e}$$

(6) *If X is a formal variable then*

$$\binom{X}{t+l} \binom{t}{w} = \textstyle\sum_v (-1)^{w-v} \binom{l+w-v-1}{w-v} \binom{X}{v} \binom{X-v}{t+l-v}. \tag{c-f}$$

Consequently, if $b + l \geqslant d + w$ then

$$\begin{aligned} &\textstyle\sum_i \binom{b-d+l}{i(p-1)+l} \binom{i(p-1)}{w} \\ &\quad = \textstyle\sum_v (-1)^{w-v} \binom{l+w-v-1}{w-v} \binom{b-d+l}{v} \mathrm{M}_{b-d+l-v,l-v}. \end{aligned} \tag{c-g}$$

(7) *We have*

$$\textstyle\sum_j (-1)^j \binom{y}{j} \binom{y+l-j}{w-j} = (-1)^w \binom{w-l-1}{w}. \tag{c-i}$$

(8) *We have*

$$\textstyle\sum_j \binom{u-1}{j-1} \binom{-l}{j-w} = (-1)^{u-w} \binom{l-w}{u-w}. \tag{c-j}$$

(9) *We have*

$$\textstyle\sum_j (-1)^j \binom{j}{b} \binom{l}{j-w} = (-1)^{l+w} \binom{w}{l+w-b}. \tag{c-k}$$

Lemma 6. *Let $\alpha \in \mathbb{Z} \cap [0, \dots, \frac{r}{p+1}]$ and let $\{D_i\}_{i \in \mathbb{Z}}$ be a family of elements of $\mathbb{Z}_p$ such that $D_i = 0$ for $i \notin [0, \frac{r-\alpha}{p-1}]$ and $\vartheta_w(D_\bullet) = 0$ for all $0 \leqslant w < \alpha$. Then*

$$\textstyle\sum_i D_i x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha} = \theta^\alpha h$$

for some polynomial h with integer coefficients.

Lemma 7. *For $\alpha, \lambda, \mu \in \mathbb{Z}_{\geqslant 0}$ let*

$$L_\alpha(\lambda, \mu)$$

be the $(\alpha+1) \times (\alpha+1)$ matrix with entries

$$L_{l,j} = \sum_{k=0}^{\alpha} \frac{j!}{l!} \left(\frac{\mu}{\lambda}\right)^k s_1(l,k) s_2(k,j),$$

where $s_1(l,k)$ are the Stirling numbers of the first kind and $s_2(k,j)$ are the Stirling numbers of the second kind. Then

$$L_\alpha(\lambda, \mu) \left(\binom{\lambda X}{0}, \dots, \binom{\lambda X}{\alpha}\right)^T = \left(\binom{\mu X}{0}, \dots, \binom{\mu X}{\alpha}\right)^T.$$

Lemma 8. *For $\alpha \in \mathbb{Z}_{\geqslant 0}$ let B_α be the $(\alpha+1) \times (\alpha+1)$ matrix with entries*

$$B_{i,j} = j! \sum_{k,l=0}^{\alpha} \frac{(-1)^{i+l+k}}{l!} \binom{l}{i} (1-p)^{-k} s_1(l,k) s_2(k,j),$$

where $s_1(i,j)$ and $s_2(k,j)$ are the Stirling numbers of the first and second kind, respectively. Let $\{X_{i,j}\}_{i,j \geqslant 0}$ be formal variables. For $\beta \in \mathbb{Z}_{\geqslant 0}$ such that $\alpha \geqslant \beta$ let

$$S(\alpha, \beta) = (S(\alpha, \beta)_{w,j})_{0 \leqslant w,j \leqslant \alpha}$$

be the $(\alpha+1) \times (\alpha+1)$ matrix with entries

$$S(\alpha, \beta)_{w,j} = \sum_{i=1}^{\beta} X_{i,j} \binom{i(p-1)}{w}.$$

Then $B_\alpha S(\alpha, \beta)$ is zero outside the rows indexed $1, \dots, \beta$ and

$$(B_\alpha S(\alpha, \beta))_{i,j} = X_{i,j}$$

for $i \in \{1, \dots, \beta\}$.

Lemma 9. *For $u, v, c \in \mathbb{Z}$ let us define*

$$F_{u,v,c}(X) = \sum_w (-1)^{w-c} \binom{w}{c} \binom{X}{w}^{\partial} \binom{X+u-w}{v-w} \in \mathbb{Q}_p[X].$$

Then

$$F_{u,v,c}(X) = \binom{u}{v-c} \binom{X}{c}^{\partial} - \binom{u}{v-c}^{\partial} \binom{X}{c}.$$

Lemma 10. *Let X and Y denote formal variables, and let*

$$c_j = (-1)^j \alpha! \left(\frac{X+j+1}{j+1} \binom{Y}{\alpha-j-1} + \binom{Y}{\alpha-j} \right) \in \mathbb{Q}[X,Y] \subset \mathbb{Q}(X,Y)$$

be polynomials over $\mathbb{Q}$ of degrees $\alpha - j$, for $1 \leqslant j \leqslant \alpha$. Let

$$M = (M_{w,j})_{0 \leqslant w,j \leqslant \alpha}$$

be the $(\alpha+1) \times (\alpha+1)$ matrix over $\mathbb{Q}(X,Y)$ with entries

$$M_{w,0} = (-1)^w \frac{(Y-X)X_w}{Y_{w+1}},$$
$$M_{w,j} = \sum_v (-1)^{w-v} \binom{j+w-v-1}{w-v} \binom{X+j}{v} \left(\binom{Y+j-v}{j-v} - \binom{X+j-v}{j-v} \right),$$

for $0 \leqslant w \leqslant \alpha$ and $0 < j \leqslant \alpha$. Then the first $\alpha - 1$ entries of

$$Mc = M(Y_\alpha, c_1, \dots, c_\alpha)^T = (d_0, \dots, d_\alpha)^T$$

are zero, and $d_\alpha = \frac{(Y-X)_{\alpha+1}}{Y-\alpha}$.

Lemma 11. *Suppose that* $s, \alpha, \beta \in \mathbb{Z}$ *are such that*

$$1 \leqslant \beta \leqslant \alpha \leqslant \tfrac{s}{2} - 2 \leqslant \tfrac{p-5}{2}.$$

Let $B = B_\alpha$ *denote the matrix defined in lemma 8. Let* M *denote the* $(\alpha+1) \times (\alpha+1)$ *matrix with entries in* $\mathbb{F}_p$ *such that if* $i \in \{1, \ldots, \beta\}$ *and* $j \in \{0, \ldots, \alpha\}$ *then*

$$M_{i,j} = \binom{\beta}{i} \cdot \begin{cases} \binom{s-\alpha-\beta+i}{i}^{-1}(-1)^{i+1} & \text{if } j = 0, \\ \binom{s-\alpha-\beta+j}{j-i} & \text{if } j > 0, \end{cases}$$

and if $i \in \{0, \ldots, \alpha\} \backslash \{1, \ldots, \beta\}$ *and* $j \in \{0, \ldots, \alpha\}$ *then* $M_{i,j}$ *is the reduction modulo* p *of*

$$p^{-[j=0]} \sum_{w=0}^{\alpha} B_{i,w} \sum_v (-1)^{w-v} \binom{j+w-v-1}{w-v} \binom{s+\beta(p-1)-\alpha+j}{v}^{\partial} \cdot \sum_{u=0}^{\beta} \binom{s+\beta(p-1)-\alpha+j-v}{u(p-1)+j-v}$$
$$- [i=0] p^{-[j=0]} \binom{s+\beta(p-1)-\alpha+j}{j}^{\partial}$$
$$- [j=0] \sum_{w=0}^{\alpha} B_{i,w} (-1)^w \binom{s+\beta(p-1)-\alpha}{w} \frac{w!}{(s-\alpha)_{w+1}}.$$

Then there is a solution of

$$M(z_0, \ldots, z_\alpha)^T = (1, 0, \ldots, 0)^T$$

such that $z_0 \neq 0$.

Now let us prove some additional combinatorial results.

Lemma 12. *Suppose that* $\alpha \in \mathbb{Z}_{\geqslant 0}$. *For* $w, j \in \{0, \ldots, \alpha\}$ *let*

$$F_{w,j}(z, \psi) \in \mathbb{F}_p[z, \psi]$$

denote the polynomial

$$\sum_v (-1)^{w-v} \binom{j+w-v-1}{w-v} \binom{z-\alpha+j}{v} \left(\binom{\psi-\alpha+j-v}{j-v} - \binom{z-\alpha+j-v}{j-v} \right).$$

Note that this depends on α. *Then*

$$\sum_{j=1}^{\alpha} (-1)^{\alpha-j} \binom{\psi-\alpha+1}{\alpha-j} F_{w,j}(z, \psi) = (-1)^\alpha ([w=\alpha] - [w=0]) \binom{\psi-z}{\alpha}.$$

Proof. Both sides of the equation we want to prove have degree α and the coefficient of z^α on each side is $\frac{1}{\alpha!}([w=\alpha] - [w=0])$. So the two sides are equal if they are equal when evaluated at the points (z, ψ) such that

$$(z, \psi) \in \{(u + \gamma(p-1) + \alpha, u + \alpha) \,|\, u \in \{0, \ldots, \alpha\}, \gamma \in \{0, \ldots, \alpha - 1\}\}.$$

The right side is zero when evaluated at these points, and

$$F_{w,j}(u + \gamma(p-1) + \alpha, u + \alpha) = \sum_{i=1}^{\gamma} \binom{u+\gamma(p-1)+j}{i(p-1)+j} \binom{i(p-1)}{w}$$

by (*c-g*). Thus we want to show that

$$\sum_{j=1}^{\alpha} (-1)^{\alpha-j} \binom{u+1}{\alpha-j} \sum_{i=1}^{\gamma} \binom{u+\gamma(p-1)+j}{i(p-1)+j} \binom{i(p-1)}{w} = \mathsf{O}(p)$$

for $0 \leqslant u, w \leqslant \alpha$ and $0 \leqslant \gamma < \alpha$. Since

$$\binom{u+\gamma(p-1)+j}{i(p-1)+j} \binom{i(p-1)}{w} = \binom{\gamma}{i} \binom{u+j-\gamma}{j-i} \binom{-i}{w} + \mathsf{O}(p),$$

that is equivalent to

$$\sum_{i,j>0} (-1)^{\alpha+w-i} \binom{u+1}{\alpha-j} \binom{\gamma}{i} \binom{\gamma-u-i-1}{j-i} \binom{i+w-1}{w} = \mathsf{O}(p).$$

This follows from the facts that

$$\sum_{j>0}\binom{u+1}{\alpha-j}\binom{\gamma-u-i-1}{j-i}=\binom{\gamma-i}{\alpha-i}$$

for $i>0$ by Vandermonde's convolution formula, and

$$\binom{\gamma}{i}\binom{\gamma-i}{\alpha-i}=\binom{\alpha}{i}\binom{\gamma}{\alpha}=0$$

since $\gamma\in\{0,\dots,\alpha-1\}$. ∎

4. Computing $\overline{\Theta}_{k,a}$

Throughout the proof we use the results from section 9 of [Ars21], which we reproduce here without proofs for convenience.

Lemma 13. *Suppose that* $\alpha\in\{0,\dots,\nu-1\}$.

(1) *We have*

$$\begin{aligned}&(T-a)\left(1\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^n x^{\alpha-n}y^{r-np-\alpha}\right)\\&\quad=\sum_j(-1)^j\binom{n}{j}p^{j(p-1)+\alpha}\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\bullet_{KZ,\overline{\mathbb{Q}}_p}x^{j(p-1)+\alpha}y^{r-j(p-1)-\alpha}\\&\qquad-a\sum_j(-1)^j\binom{n-\alpha}{j}\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^\alpha x^{j(p-1)}y^{r-j(p-1)-\alpha(p+1)}+\mathsf{O}(p^n).\end{aligned}$$

(2) *The submodule* $\operatorname{im}(T-a)\subset\operatorname{ind}_{KZ}^G\widetilde{\Sigma}_r$ *contains*

$$\sum_i\left(\sum_{l=\beta-\gamma}^{\beta}C_l\binom{r-\beta+l}{i(p-1)+l}\right)\bullet_{KZ,\overline{\mathbb{Q}}_p}x^{i(p-1)+\beta}y^{r-i(p-1)-\beta}+\mathsf{O}\left(ap^{-\beta+v_C}+p^{p-1}\right)$$

for all $0\leqslant\beta\leqslant\gamma<\nu$ *and all families* $\{C_l\}_{l\in\mathbb{Z}}$ *of elements of* $\mathbb{Z}_p$, *where*

$$v_C=\min_{\beta-\gamma\leqslant l\leqslant\beta}(v_p(C_l)+l).$$

The $\mathsf{O}\left(ap^{-\beta+v_C}+p^{p-1}\right)$ *term is equal to* $\mathsf{O}\left(p^{p-1}\right)$ *plus*

$$-\frac{ap^{-\beta}}{p-1}\sum_{l=\beta-\gamma}^{\beta}C_lp^l\sum_{0\neq\mu\in\mathbb{F}_p}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^n x^{\beta-l-n}y^{r-np-\beta+l}.$$

Lemma 14. *Suppose that* $\alpha\in\mathbb{Z}$ *and* $v\in\mathbb{Q}$ *are such that*

$$\begin{aligned}&\alpha\in\{0,\dots,\nu-1\},\\&v\leqslant v_p(\vartheta_\alpha(D_\bullet)),\\v':=\min\{v_p(a)-\alpha,v\}&\leqslant v_p(\vartheta_w(D_\bullet))\text{ for }\alpha<w<2\nu-\alpha,\\&v'<v_p(\vartheta_w(D_\bullet))\text{ for }0\leqslant w<\alpha.\end{aligned}$$

If, for $j\in\mathbb{Z}$,

$$\Delta_j:=(-1)^{j-1}(1-p)^{-\alpha}\binom{\alpha}{j-1}\vartheta_\alpha(D_\bullet),$$

then $v\leqslant v_p(\vartheta_\alpha(\Delta_\bullet))\leqslant v_p(\Delta_j)$ *for all* $j\in\mathbb{Z}$, *and*

$$\begin{aligned}&\sum_i(\Delta_i-D_i)\bullet_{KZ,\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{r-i(p-1)-\alpha}\\&\quad=[\alpha\leqslant s](-1)^{n+1}D_{\frac{r-s}{p-1}}\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^n x^{r-np-s+\alpha}y^{s-\alpha-n}\\&\qquad-D_0\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^n x^{\alpha-n}y^{r-np-\alpha}\\&\qquad\quad+E\bullet_{KZ,\overline{\mathbb{Q}}_p}\theta^{\alpha+1}h+F\bullet_{KZ,\overline{\mathbb{Q}}_p}h'+\mathrm{ERR}_1+\mathrm{ERR}_2,\end{aligned}$$

for some ERR_1 *and* ERR_2 *such that*

$$\mathrm{ERR}_1\in\operatorname{im}(T-a)\text{ and }\mathrm{ERR}_2=\mathsf{O}\left(p^{\nu-v_p(a)+v}+p^{\nu-\alpha}\right),$$

some polynomials h, h', and some $E, F \in \overline{\mathbb{Q}}_p$ such that $v_p(E) \geqslant v'$ and $v_p(F) > v'$.

Lemma 15. *Let $\{C_l\}_{l\in\mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$. Suppose that*

$$\alpha \in \{0, \dots, \nu - 1\}$$

and $v \in \mathbb{Q}$, and suppose that the constants

$$D_i := [i = 0]C_{-1} + [0 < i(p-1) < r - 2\alpha] \textstyle\sum_{l=0}^{\alpha} C_l \binom{r-\alpha+l}{i(p-1)+l}$$

satisfy the conditions of lemma 14, i.e.

$$v \leqslant v_p(\vartheta_\alpha(D_\bullet)),$$
$$v' := \min\{v_p(a) - \alpha, v\} \leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha < w < 2\nu - \alpha,$$
$$v' < v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha.$$

Moreover, suppose that C_0 is a unit. Let

$$\vartheta' := (1-p)^{-\alpha}\vartheta_\alpha(D_\bullet) - C_{-1}.$$

Suppose that $v_p(C_{-1}) \geqslant v_p(\vartheta')$.

(1) *If $v_p(\vartheta') \leqslant v'$ then there is some element $\mathrm{gen}_1 \in \mathscr{I}_a$ that represents a generator of $\widehat{N}_\alpha$.*

(2) *If $v_p(a) - \alpha < v$ then there is some element $\mathrm{gen}_2 \in \mathscr{I}_a$ that represents a generator of a finite-codimensional submodule of*

$$T\left(\mathrm{ind}_{KZ}^G \mathrm{quot}(\alpha)\right) = T\left(\widehat{N}_\alpha / \mathrm{ind}_{KZ}^G \mathrm{sub}(\alpha)\right),$$

where T denotes the endomorphism of $\mathrm{ind}_{KZ}^G \mathrm{quot}(\alpha)$ corresponding to the double coset of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let us now prove the following additional results.

Lemma 16. *Let $\{C_l\}_{l\in\mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$. Suppose that $\alpha \in \mathbb{Z}$ and $v \in \mathbb{Q}$ and the constants*

$$D_i := [i = 0]C_{-1} + [0 < i(p-1) < r - 2\alpha] \textstyle\sum_{l=0}^{\alpha} C_l \binom{r-\alpha+l}{i(p-1)+l}$$

are such that

$$\alpha \in \{0, \dots, \nu - 1\},$$
$$v \leqslant v_p(\vartheta_\alpha(D_\bullet)),$$
$$v' := \min\{v_p(a) - \alpha, v\} < v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha.$$

Let

$$\vartheta' := (1-p)^{-\alpha}\vartheta_\alpha(D_\bullet) - C_{-1}.$$

Then $\mathrm{im}(T - a)$ contains

$$(\vartheta' + C_{-1}) \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\alpha x^{p-1} y^{r-\alpha(p+1)-p+1} + C_{-1} \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^n x^{\alpha-n} y^{r-np-\alpha}$$
$$+ \textstyle\sum_{\xi=\alpha+1}^{2\nu-\alpha-1} E_\xi \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\xi h_\xi + F \bullet_{KZ,\overline{\mathbb{Q}}_p} h' + H, \tag{1}$$

for some h_ξ, h', E_ξ, F, H such that

(1) *$E_\xi = \vartheta_\xi(D_\bullet) + \mathsf{O}(p^v) \cup \mathsf{O}(\vartheta_{\alpha+1}(D_\bullet)) \cup \cdots \cup \mathsf{O}(\vartheta_{\xi-1}(D_\bullet))$,*

(2) *if* $\underline{\xi+\alpha-s} \leqslant \underline{2\xi-s} \neq 0$ *then the reduction modulo* $\mathfrak{m}$ *of* $\theta^\xi h_\xi$ *generates* N_ξ,

(3) $v_p(F) > v'$, *and*

(4) $H = \mathsf{O}\big(p^{\nu-v_p(a)+v} + p^{\nu-\alpha}\big)$ *and if* $v_p(a)-\alpha < v$ *then*

$$\tfrac{1-p}{ap^{-\alpha}} H = g \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^n x^{\alpha-n} y^{r-np-\alpha} + \mathsf{O}\big(p^{\nu-v_p(a)}\big)$$

with

$$g = \textstyle\sum_{\lambda\in\mathbb{F}_p} C_0\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) + A\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + [r \equiv_{p-1} 2\alpha] B\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right),$$

where

$$A = -C_{-1} + \textstyle\sum_{l=1}^{\alpha} C_l \binom{r-\alpha+l}{l}$$

and

$$B = \textstyle\sum_{l=0}^{\alpha} C_l \binom{r-\alpha+l}{s-\alpha}.$$

Proof. This lemma is essentially shown under a stronger hypothesis as lemma 15. The stronger hypothesis consists of the three extra conditions that

$$v_p(\vartheta_w(D_\bullet)) \geqslant \min\{v_p(a)-\alpha, v\}$$

for all $\alpha < w < 2\nu-\alpha$, that $C_0 \in \mathbb{Z}_p^\times$, and that $v_p(C_{-1}) \geqslant v_p(\vartheta')$. These extra conditions are not used in the actual construction of the element in (1), rather they are there to ensure that $v_p(E_\xi) \geqslant \min\{v_p(a)-\alpha, v\}$ for all $\alpha < \xi < 2\nu-\alpha$, that the coefficient of $\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right)$ in g is invertible, and that we get an integral element once we divide the element

$$(\vartheta' + C_{-1}) \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\alpha x^{p-1} y^{r-\alpha(p+1)-p+1} + C_{-1} \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^n x^{\alpha-n} y^{r-np-\alpha}$$

by ϑ'. Therefore we still get the existence of the element in (1) without these extra conditions, and to complete the proof of lemma 16 we need to verify the properties of h_ξ, E_ξ, F, H, A, and B claimed in (1), (2), (3), and (4). The h_ξ and E_ξ come from the proof of lemma 14, and $E_\xi \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\xi h_\xi$ is

$$X_\xi \bullet_{KZ,\overline{\mathbb{Q}}_p} \textstyle\sum_{\lambda\neq 0} [-\lambda]^{r-\alpha-\xi} \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) (-\theta)^n x^{r-np-\xi} y^{\xi-n},$$

with the notation for X_ξ from the proof of lemma 14. Let $E_\xi = (-1)^{\xi+1} X_\xi$. Then condition (1) is satisfied directly from the definition of X_ξ. Let

$$h_\xi = (-1)^{\xi+1} \textstyle\sum_{\lambda\neq 0} [-\lambda]^{r-\alpha-\xi} \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) (-\theta)^n x^{r-np-\xi} y^{\xi-n}.$$

This reduces modulo $\mathfrak{m}$ to the element

$$(-1)^\xi \textstyle\sum_{\lambda\neq 0} [-\lambda]^{r-\alpha-\xi} \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) Y^{\underline{2\xi-r}} = (-1)^{s-\alpha+1} \binom{\underline{2\xi-s}}{\underline{\xi+\alpha-s}} X^{\underline{\xi+\alpha-s}} Y^{\underline{\xi-\alpha}}$$

of

$$\sigma_{\underline{2\xi-r}}(r-\xi) \cong I_{r-2\xi}(\xi)/\sigma_{\overline{r-2\xi}}(\xi) = \mathrm{quot}(\xi).$$

This element is non-trivial and generates N_ξ if $\underline{\xi+\alpha-s} \leqslant \underline{2\xi-s} \neq 0$, since then $X^{\underline{\xi+\alpha-s}} Y^{\underline{\xi-\alpha}}$ generates N_ξ. This verifies condition (2). Condition (3) follows from the assumption $v' < v_p(\vartheta_w(D_\bullet))$ for $0 \leqslant w < \alpha$, as in the proof of lemma 14. Finally, condition (4) follows from the description of the error term in lemma 13, as in the proof of lemma 15. ∎

Corollary 17. *Let $\{C_l\}_{l\in\mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$. Suppose that*

$$\alpha \in \{0, \dots, \nu - 1\}$$

and $v \in \mathbb{Q}$, and suppose that the constants

$$D_i := [i = 0]C_{-1} + [0 < i(p-1) < r - 2\alpha] \textstyle\sum_{l=0}^{\alpha} C_l \binom{r-\alpha+l}{i(p-1)+l}$$

are such that

$$\begin{aligned} v &\leqslant v_p(\vartheta_\alpha(D_\bullet)), \\ v' := \min\{v_p(a) - \alpha, v\} &\leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha < w < 2\nu - \alpha, \\ v' &< v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha. \end{aligned}$$

Suppose also that $v_p(a) \notin \mathbb{Z}$. Let

$$\begin{aligned} \vartheta' &:= (1-p)^{-\alpha}\vartheta_\alpha(D_\bullet) - C_{-1}, \\ \check{C} &:= -C_{-1} + \textstyle\sum_{l=1}^{\alpha} C_l \binom{r-\alpha+l}{l}. \end{aligned}$$

If $\star$ then $*$ is trivial modulo $\mathscr{I}_a$, for each of the following pairs

$$(\star, *) = (\textit{condition}, \textit{representation}).$$

(1) $\Big(v_p(\vartheta') \leqslant \min\{v_p(C_{-1}), v'\},\ \widehat{N}_\alpha\Big)$.

(2) $\Big(v = v_p(C_{-1}) < \min\{v_p(\vartheta'), v_p(a) - \alpha\},\ \operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha)\Big)$.

(3) $\Big(v_p(a) - \alpha < v \leqslant v_p(C_{-1})\ \&\ \check{C} \in \mathbb{Z}_p^\times\ \&\ C_0 \notin \mathbb{Z}_p^\times\ \&\ \underline{2\alpha - r} > 0,\ \widehat{N}_\alpha\Big)$.

(4) $\Big(v_p(a) - \alpha < v \leqslant v_p(C_{-1})\ \&\ \check{C} \in \mathbb{Z}_p^\times,\ \operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)\Big)$.

(5) $\Big(v_p(a) - \alpha < v \leqslant v_p(C_{-1})\ \&\ C_0 \in \mathbb{Z}_p^\times,\ \mathbf{r}_1\Big)$, *where*

$$\mathbf{r}_1$$

is a finite-codimensional submodule of

$$T(\operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)).$$

Proof. There is one extra condition imposed in addition to the conditions from lemma 16: that

$$v' := \min\{v_p(a) - \alpha, v\} \leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha < w < 2\nu - \alpha,$$

and it ensures that $v_p(E_\xi) \geqslant v'$ for all $\alpha < \xi < 2\nu - \alpha$. Lemma 16 implies that the element in (1) is in $\operatorname{im}(T - a)$. Let us call this element γ.

(1) The condition $v_p(\vartheta') \leqslant \min\{v_p(C_{-1}), v'\}$ ensures that if we divide γ by ϑ' then the resulting element reduces modulo $\mathfrak{m}$ to a representative of a generator of $\widehat{N}_\alpha$.

(2) The condition $v = v_p(C_{-1}) < \min\{v_p(\vartheta'), v_p(a) - \alpha\}$ ensures that if we divide γ by C_{-1} then the resulting element reduces modulo $\mathfrak{m}$ to a representative of a generator of $\operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha)$.

(3, 4, 5) The condition $v_p(a)-\alpha < v \leqslant v_p(C_{-1})$ ensures that the term with the dominant valuation in (1) is H, so we can divide γ by $ap^{-\alpha}$ and obtain the element $L+\mathsf{O}\big(p^{\nu-v_p(a)}\big)$, where L is defined by

$$L := \Big(\textstyle\sum_{\lambda\in\mathbb{F}_p} C_0\big(\begin{smallmatrix} p & [\lambda] \\ 0 & 1\end{smallmatrix}\big) + A\big(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\big) + [r \equiv_{p-1} 2\alpha] B\big(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\big)\Big) \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^n x^{\alpha-n} y^{r-np-\alpha}$$

with A and B as in lemma 16. This element L is in $\operatorname{im}(T-a)$, and it reduces modulo $\mathfrak{m}$ to a representative of

$$\Big(\textstyle\sum_{\lambda\in\mathbb{F}_p} C_0\big(\begin{smallmatrix} p & [\lambda] \\ 0 & 1\end{smallmatrix}\big) + A\big(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\big) + [r \equiv_{p-1} 2\alpha](-1)^{r-\alpha} B\big(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\big)\Big) \bullet_{KZ,\overline{\mathbb{F}}_p} X^{\underline{2\alpha-r}}.$$

As shown in the proof of lemma 15, if $C_0 \in \mathbb{Z}_p^\times$ then this element always generates a finite-codimensional submodule of

$$T(\operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)),$$

and if additionally $A \neq 0$ (over $\mathbb{F}_p$) then in fact we have the stronger conclusion that it generates

$$\operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha).$$

Suppose on the other hand that $C_0 = \mathsf{O}(p)$ and $A \in \mathbb{Z}_p^\times$. In that case we assume that $\underline{2\alpha - r} > 0$ and therefore the reduction modulo $\mathfrak{m}$ of L represents a generator of $\widehat{N}_\alpha$. ∎

Corollary 18. *Let $\{C_l\}_{l\in\mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$. Suppose that*

$$\alpha \in \{0,\ldots,\nu-1\}$$

and $v\in\mathbb{Q}$, and suppose that the constants

$$D_i := [i=0]C_{-1} + [0 < i(p-1) < r-2\alpha] \textstyle\sum_{l=0}^{\alpha} C_l \binom{r-\alpha+l}{i(p-1)+l}$$

are such that

$$v \leqslant v_p(\vartheta_\alpha(D_\bullet)),$$
$$v' := \min\{v_p(a)-\alpha, v\} \leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha < w < 2\nu-\alpha,$$
$$v' < v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha.$$

Suppose also that $v_p(a)\in\mathbb{Z}$. Let

$$\vartheta' := (1-p)^{-\alpha}\vartheta_\alpha(D_\bullet) - C_{-1},$$
$$\check{C} := -C_{-1} + \textstyle\sum_{l=1}^{\alpha} C_l \binom{r-\alpha+l}{l}.$$

If $\star$ then $*$ is trivial modulo $\mathscr{I}_a$, for each of the following pairs

$$(\star, *) = (\textit{condition}, \textit{representation}).$$

(1) $\Big(v_p(\vartheta') \leqslant \min\{v_p(C_{-1}), v\}$ *&* $v_p(\vartheta') < v_p(a)-\alpha,\ \widehat{N}_\alpha\Big)$.

(2) $\Big(v = v_p(C_{-1}) < \min\{v_p(\vartheta'), v_p(a)-\alpha\},\ \operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha)\Big)$.

(3) $\Big(v_p(a)-\alpha < v \leqslant v_p(C_{-1})$ *&* $\check{C}\in\mathbb{Z}_p^\times$ *&* $C_0 \notin \mathbb{Z}_p^\times$ *&* $\underline{2\alpha-r} > 0,\ \widehat{N}_\alpha\Big)$.

(4) $\Big(v_p(a)-\alpha < v \leqslant v_p(C_{-1})$ *&* $\check{C}\in\mathbb{Z}_p^\times,\ \operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)\Big)$.

(5) $\Big(v_p(a) - \alpha < v \leqslant v_p(C_{-1})$ *&* $C_0 \in \mathbb{Z}_p^\times,\, \mathbf{r}_2\Big)$, *where*

$$\mathbf{r}_2$$

is a finite-codimensional submodule of

$$T(\mathrm{ind}_{KZ}^G \operatorname{quot}(\alpha)).$$

(6) $\Big(v_p(a) - \alpha = v = v_p(\vartheta') \leqslant v_p(C_{-1})$ *&* $\check{C} \notin \mathbb{Z}_p^\times$ *&* $C_0 \in \mathbb{Z}_p^\times,\, \mathbf{r}_3\Big)$ *for*

$$\mathbf{r}_3 = \Big(T + \hat{C}\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right) - \tfrac{C_0^{-1}\vartheta'}{ap^{-\alpha}}\Big)\,(\mathrm{ind}_{KZ}^G \operatorname{quot}(\alpha)),$$

where

$$\hat{C} = [r \equiv_{p-1} 2\alpha]\,\Big((-1)^\alpha \textstyle\sum_{l=0}^{\alpha} C_0^{-1} C_l \binom{r-\alpha+l}{s-\alpha} - 1\Big).$$

(7) $\Big(v_p(a) - \alpha = v = v_p(C_{-1}) < v_p(\vartheta')$ *&* $C_0 \notin \mathbb{Z}_p^\times$ *&* $\underline{2\alpha - r} > 0,\, \mathbf{r}_4\Big)$ *for*

$$\mathbf{r}_4 = \Big(T + \tfrac{\check{C}ap^{-\alpha}}{C_{-1}}\Big)\,(\mathrm{ind}_{KZ}^G \operatorname{sub}(\alpha)).$$

(8) $\Big(v_p(a) - \alpha = v = v_p(C_{-1}) < v_p(\vartheta')$ *&* $\check{C} \in \mathbb{Z}_p^\times,\, \mathrm{ind}_{KZ}^G \operatorname{quot}(\alpha)\Big)$

(9) $\Big(v_p(a) - \alpha = v = v_p(C_{-1}) < v_p(\vartheta')$ *&* $C_0 \in \mathbb{Z}_p^\times,\, \mathbf{r}_5\Big)$, *where*

$$\mathbf{r}_5$$

is a finite-codimensional submodule of

$$T(\mathrm{ind}_{KZ}^G \operatorname{quot}(\alpha)).$$

Proof. (1, 2, 3, 4, 5) The proofs of these parts are nearly identical to the proofs of the corresponding parts of corollary 17.

(6) The proof is similar to the proof of (5), the only difference being that the valuation of ϑ' is the same as the valuation of the coefficient of H. To be more specific, we divide γ by C_0, the term "T" comes from the expression for H given in lemma 16, the term "$\hat{C}\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)$" comes from

$$[r \equiv_{p-1} 2\alpha](C_0^{-1}B - 1)\left(\begin{smallmatrix}0 & 1\\ p & 0\end{smallmatrix}\right),$$

the reason there is no term "$A\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right)$" is because $A = \check{C} = \mathsf{O}(p)$, and the term "$-\frac{C_0^{-1}\vartheta'}{ap^{-\alpha}}$" comes from the first line of the formula for γ given in (1).

(7) As in the previous parts we can deduce that $\mathscr{I}_a$ contains

$$L := \tfrac{\check{C}ap^{-\alpha}}{C_{-1}}\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right) \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\alpha(y^{r'} - x^{p-1}y^{r'-p+1}) + 1 \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\alpha y^{r'} + L',$$

where $r' = r - \alpha(p+1)$ and L' reduces modulo $\mathfrak{m}$ to a trivial element of $\operatorname{sub}(\alpha)$. The reduction modulo $\mathfrak{m}$ of the element $\sum_{\mu\in\mathbb{F}_p}\left(\begin{smallmatrix}1 & 0\\ [\mu] & p\end{smallmatrix}\right)L$ generates $\mathbf{r}_4$.

(8, 9) The proofs of these parts are similar to the proofs of (4, 5). ∎

5. Proof of theorem 3

The proof of theorem 3 is based on the approach outlined in [BG09], and roughly consists of finding enough elements in $\mathscr{I}_a$, consequently eliminating enough subquotients of $\operatorname{ind}_{KZ}^G \Sigma_r$, and using that information to find $\overline{\Theta}_{k,a}$.

Throughout this section we assume that

$$r = s + \beta(p-1) + u_0 p^t + \mathsf{O}\big(p^{t+1}\big)$$

for some $\beta \in \{0, \dots, p-1\}$ and $u_0 \in \mathbb{Z}_p^\times$ and $t \in \mathbb{Z}_{>0}$. Let us write $\eta = u_0 p^t$. Recall also that we assume $\nu - 1 < v_p(a) < \nu$ for some $\nu \in \{1, \dots, \frac{p-1}{2}\}$, that

$$s \in \{2\nu, \dots, p-2\},$$

and that $k > p^{100}$ (and consequently $r > p^{99}$).

Let us first show the equivalence between theorem 3 and the union of the following two propositions.

Proposition 19. *If $k \in \mathscr{R}_0^{s,\nu}$ then any infinite-dimensional factor of $\overline{\Theta}_{k,a}$ is a quotient of $\widehat{N}_{\nu-1}$. If $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ then none of the infinite-dimensional factors of $\overline{\Theta}_{k,a}$ are quotients of a representation in the set*

$$\left\{\widehat{N}_0, \dots, \widehat{N}_{\max\{\nu-t-1,\beta\}-1}\right\}.$$

Proposition 20. *If $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ then none of the infinite-dimensional factors of $\overline{\Theta}_{k,a}$ are quotients of a representation in the set*

$$\left\{\widehat{N}_{\max\{\nu-t-1,\beta\}+1}, \dots, \widehat{N}_{\nu-1}\right\}.$$

Proof that theorem 3 is equivalent to propositions 19 + 20. First let us assume that propositions 19 and 20 are true. Together they imply that any infinite-dimensional factor of $\overline{\Theta}_{k,a}$ is a quotient of $\widehat{N}_\alpha$, where $\alpha = \nu - 1$ if $k \in \mathscr{R}_0^{s,\nu}$ and $\alpha = \max\{\nu - t - 1, \beta\}$ if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$. The classifications given by theorem 2 in [Ars21] and theorem 2.7.1 in [Bre03a] imply that if $\overline{\Theta}_{k,a}$ is reducible then it must have exactly two infinite-dimensional factors. There may be an additional one-dimensional factor, a twist of the Steinberg representation. Suppose that the infinite-dimensional factors are quotients of $\operatorname{ind}_{KZ}^G(\sigma_b(b'))$ and $\operatorname{ind}_{KZ}^G(\sigma_d(d'))$, respectively. By theorem 2 in [Ars21] we must have

$$b' - d' \equiv_{p-1} d + 1 \text{ and } b + d \equiv_{p-1} -2.$$

In particular, $\operatorname{ind}_{KZ}^G(\sigma_b(b')), \operatorname{ind}_{KZ}^G(\sigma_d(d'))$ cannot be $\operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha), \operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)$, as that would imply that

$$2\alpha - r \equiv_{p-1} 2\alpha - r + 1.$$

Similarly, the two representations cannot be two copies of $\operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha)$, as that would imply that

$$2\alpha \equiv_{p-1} s + 1,$$

a contradiction since $2\alpha \in \{2, \dots, 2\nu - 2\}$ and $s \in \{2\nu, \dots, p-1\}$. And, the two representations cannot be two copies of $\operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)$, as that would imply that

$$2\alpha \equiv_{p-1} s - 1,$$

which is similarly a contradiction. Thus we can conclude that $\overline{\Theta}_{k,a}$ must be irreducible, and the classifications given by theorem 2 in [Ars21] and theorem 2.7.1 in [Bre03a] imply that the only possible quotient of $\widehat{N}_\alpha$ that $\overline{\Theta}_{k,a}$ can be is $\mathrm{BIrr}(b_\alpha)$. This implies theorem 3.

Conversely, if theorem 3 is true, then the fact that $\overline{\Theta}_{k,a} \cong \mathrm{BIrr}(b_\alpha)$ implies that $\overline{\Theta}_{k,a}$ is irreducible and not a quotient of a representation in the set

$$\left\{\widehat{N}_0, \ldots, \widehat{N}_{\nu-1}\right\} \setminus \{N_\alpha\}.$$

∎

Proposition 19 is essentially the main result of [Ars21]. Let us now prove proposition 20.

Proof of proposition 20. Let $\alpha \in \{0, \ldots, \nu-2\}$ and let us consider $\widehat{N}_\alpha$. The task is to show that if $\alpha > \max\{\nu - t - 1, \beta\}$ then none of the infinite-dimensional factors of $\overline{\Theta}_{k,a}$ are quotients of $\widehat{N}_\alpha$. Note that the condition on α implies both $\alpha > \beta$ and

$$t \geqslant \nu - \alpha > v_p(a) - \alpha.$$

Let us apply part (3) of corollary 17 with v chosen arbitrarily in the open interval $(v_p(a) - \alpha, t)$ and

$$C_j = \begin{cases} 0 & \text{if } j \in \{-1, 0\}, \\ (-1)^{\alpha-j}\binom{s-\alpha+1}{\alpha-j} + pC_j^* & \text{if } j \in \{1, \ldots, \alpha\}, \end{cases}$$

for some constants $C_1^*, \ldots, C_\alpha^*$ yet to be chosen. We need to show that the constants $\{C_j\}$ are suitable, i.e. that the conditions of corollary 17 are satisfied. Clearly

$$v_p(a) - \alpha < v \leqslant v_p(C_{-1})$$

and $C_0 = \mathsf{O}(p)$. Moreover,

$$\begin{aligned} \textstyle\sum_{l=1}^{\alpha} C_l \binom{r-\alpha+l}{l} &= \textstyle\sum_{l=1}^{\alpha} (-1)^{\alpha-l}\binom{s-\alpha+1}{\alpha-l}\binom{s-\alpha-\beta+l}{l} + \mathsf{O}(p) \\ &= (-1)^{\alpha}\binom{\beta}{\alpha} + (-1)^{\alpha+1}\binom{s-\alpha+1}{\alpha} + \mathsf{O}(p) \\ &= (-1)^{\alpha+1}\binom{s-\alpha+1}{\alpha} + \mathsf{O}(p). \end{aligned}$$

The third equality follows from the fact that $\alpha > \beta$. Since

$$p + \alpha - 1 > s > 2\alpha - 2$$

for $\alpha > 0$, and $\binom{s-\alpha+1}{\alpha} = 1$ for $\alpha = 0$, it follows that

$$\textstyle\sum_{l=1}^{\alpha} C_l \binom{r-\alpha+l}{l} \in \mathbb{Z}_p^\times.$$

Thus we only need to verify the most delicate condition, that

$$v \leqslant v_p(\vartheta_w(D_\bullet))$$

for $0 \leqslant w < 2\nu - \alpha$. By (*c-a*) and (*c-g*), if

$$L_1(r) := \textstyle\sum_{i>0} \binom{r-\alpha+j}{i(p-1)+j}\binom{i(p-1)}{w}$$

then $L_1(r) = L_1(s + \beta(p-1)) + \mathsf{O}(\eta)$ for $0 \leqslant w < 2\nu - \alpha$. So in order to verify the last condition it is enough to show that

$$L_2(s) := \textstyle\sum_{j=1}^{\alpha} \left((-1)^{\alpha-j}\binom{s-\alpha+1}{\alpha-j} + pC_j^*\right)\binom{s+\beta(p-1)-\alpha+j}{i(p-1)+j} = \mathsf{O}(p^v) \tag{2}$$

for all $i \in \{1, \dots, \beta\}$. We have

$$\tbinom{s+\beta(p-1)-\alpha+j}{i(p-1)+j} = \tbinom{\beta}{i}\tbinom{s-\alpha-\beta+j}{j-i} + \mathsf{O}(p).$$

We also have

$$\begin{aligned}
&\textstyle\sum_{j=1}^{\alpha}(-1)^{\alpha-j}\binom{s-\alpha+1}{\alpha-j}\binom{s-\alpha-\beta+j}{j-i}\\
&\quad= \textstyle\sum_{j=1}^{\alpha}(-1)^{\alpha-i}\binom{s-\alpha+1}{\alpha-j}\binom{\alpha+\beta-s-i-1}{j-i}\\
&\quad= (-1)^{\alpha-i}\left(-\binom{s-\alpha+1}{\alpha}\binom{\alpha+\beta-s-i-1}{-i} + \textstyle\sum_j \binom{s-\alpha+1}{\alpha-j}\binom{\alpha+\beta-s-i-1}{j-i}\right)\\
&\quad= (-1)^{\alpha-i}\left(-[i=0]\binom{s-\alpha+1}{\alpha} + \binom{\beta-i}{\alpha-i}\right) = 0.
\end{aligned}$$

The second equality follows from Vandermonde's convolution formula. The third equality follows from the assumptions that $i > 0$ and $\alpha > \beta$. So (2) is true modulo p, and we can transform (2) into the matrix equation

$$\left(\tbinom{s+\beta(p-1)-\alpha+j}{i(p-1)+j}\right)_{1\leqslant i\leqslant\beta,\, 1\leqslant j\leqslant\alpha}(C_1^*, \dots, C_\alpha^*)^T = (w_1, \dots, w_\beta)^T$$

for some $w_1, \dots, w_\beta \in \mathbb{Z}_p$. This matrix equation always has a solution since the left $\beta\times\beta$ submatrix of the reduction modulo p of the matrix

$$\left(\tbinom{s+\beta(p-1)-\alpha+j}{i(p-1)+j}\right)_{1\leqslant i\leqslant\beta,\, 1\leqslant j\leqslant\alpha}$$

is upper triangular with units on the diagonal. Therefore we can indeed always choose the constants $C_1^*, \dots, C_\alpha^*$ in a way that $v \leqslant v_p(\vartheta_w(D_\bullet))$ for $0 \leqslant w < 2\nu - \alpha$. Then all conditions of part (3) of corollary 17 are satisfied, which concludes the proof of proposition 20. ∎

Propositions 19 and 20 are already sufficient to compute $\overline{\Theta}_{k,a}$, since they imply that $\overline{\Theta}_{k,a}$ must be a quotient of $\widehat{N}_\alpha$, where $\alpha = \nu - 1$ if $k \in \mathscr{R}_0^{s,\nu}$ and $\alpha = \max\{\nu - t - 1, \beta\}$ if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$. Let us show that in fact the surjective map

$$\widehat{N}_\alpha \longrightarrow\!\!\!\!\!\rightarrow \overline{\Theta}_{k,a}$$

factors through $\operatorname{ind}_{KZ}^G \operatorname{quot}(\alpha)$.

Proposition 21. *If $k \in \mathscr{R}_0^{s,\nu}$ then the surjective map*

$$\widehat{N}_{\nu-1} \longrightarrow\!\!\!\!\!\rightarrow \overline{\Theta}_{k,a}$$

factors through $\operatorname{ind}_{KZ}^G \operatorname{quot}(\nu - 1)$.

Proof. Let $\alpha = \nu - 1$. Note that since $k \in \mathscr{R}_0^{s,\nu}$ we have

$$\beta \in \{\nu - 1, \dots, p - 1\}.$$

Let us apply part (2) of corollary 17 with $v = 0$ and some constants

$$C_{-1}, C_0, \dots, C_\alpha$$

such that $C_{-1} = \binom{\beta}{\alpha}$ and $C_0 = 0$. The conditions that need to be satisfied in order for the corollary to be applicable are $v_p(\vartheta_w(D_\bullet)) > 0$ for $0 \leqslant w < \alpha$, and $v_p(\vartheta') > 0$. Let us consider the matrix $A = (A_{w,j})_{0\leqslant w,j\leqslant\alpha}$ that has integer entries

$$A_{w,j} = \textstyle\sum_{i>0}\binom{r-\alpha+j}{i(p-1)+j}\binom{i(p-1)}{w}.$$

Then $v_p(\vartheta') > 0$ is equivalent to $\vartheta_\alpha(D_\bullet) = C_{-1} + \mathsf{O}(p) = \binom{\beta}{\alpha} + \mathsf{O}(p)$, so the two equations we want to show are equivalent to

$$A(0, C_1, \dots, C_\alpha)^T = \left(-\tbinom{\beta}{\alpha}, 0, \dots, 0, \tbinom{\beta}{\alpha}\right)^T + \mathsf{O}(p).$$

By (*c*-*g*) we have

$$\overline{A}_{w,j} = F_{w,j}(r,s),$$

where $F_{w,j}(z,\psi) \in \mathbb{F}_p[z,\psi]$ is the polynomial defined in lemma 12. Then the conclusion of that lemma is that

$$\overline{A}(0, C_1, \dots, C_\alpha)^T = \left(-\tbinom{\beta}{\alpha}, 0, \dots, 0, \tbinom{\beta}{\alpha}\right)^T$$

with

$$C_j = (-1)^j \tbinom{s-\alpha+1}{\alpha-j}$$

for $j \in \{1, \dots, \alpha\}$. Thus these choices for $C_{-1}, C_0, \dots, C_\alpha$ are suitable, and we can apply part (2) of corollary 17 with $v = 0$ and conclude that $\operatorname{ind}_{KZ}^G \operatorname{sub}(\alpha)$ is trivial modulo $\mathscr{I}_a$. ∎

Proposition 22. *If* $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ *then the surjective map*

$$\widehat{N}_{\max\{\nu-t-1,\beta\}} \longrightarrow\!\!\!\!\!\rightarrow \overline{\Theta}_{k,a}$$

factors through $\operatorname{ind}_{KZ}^G \operatorname{quot}(\max\{\nu-t-1,\beta\})$.

Proof. Let $\alpha = \max\{\nu - t - 1, \beta\}$. Let us apply part (2) of corollary 17 with $v = t$ and

$$C_j = \begin{cases} \eta & \text{if } j = -1, \\ 0 & \text{if } j = 0, \\ (-1)^{\alpha+\beta+j}\alpha\binom{\alpha-1}{\beta}\binom{s-\alpha+1}{\alpha-j} & \text{if } j \in \{1, \dots, \alpha\}. \end{cases}$$

We need to show that the constants $\{C_j\}$ are suitable, i.e. that the conditions of corollary 17 are satisfied. Clearly $v_p(C_{-1}) = t < v_p(a) - \alpha$. We also need to show that $t < v_p(\vartheta_w(D_\bullet))$ for $0 \leqslant w < \alpha$ and $t < v_p(\vartheta')$ and $t \leqslant v_p(\vartheta_w(D_\bullet))$ for $\alpha \leqslant w < 2\nu - \alpha$. Let us consider the matrix $A = (A_{w,j})_{0 \leqslant w,j \leqslant \alpha}$ that has integer entries

$$A_{w,j} = \textstyle\sum_{i>0} \binom{r-\alpha+j}{i(p-1)+j}\binom{i(p-1)}{w}.$$

If we consider the approximation claim in the proof of proposition 19 and multiply the first column by p, we get

$$A = S + \eta N + \mathsf{O}(\eta p),$$

where

$$\begin{aligned} S_{w,j} &= \textstyle\sum_{i=1}^{\beta} \binom{s+\beta(p-1)-\alpha+j}{i(p-1)+j}\binom{i(p-1)}{w}, \\ N_{w,j} &= \textstyle\sum_v (-1)^{w-v}\binom{j+w-v-1}{w-v}\binom{s+\beta(p-1)-\alpha+j}{v}^{\partial} \sum_{i=0}^{\beta} \binom{s+\beta(p-1)-\alpha+j-v}{i(p-1)+j-v} \\ &\quad - [w=0]\binom{s+\beta(p-1)-\alpha+j}{j}^{\partial}. \end{aligned}$$

Exactly as in the proof of proposition 19 we can deduce the three conditions we need to show as long as

$$S(C_0,\dots,C_\alpha)^T = 0,$$
$$N(C_0,\dots,C_\alpha)^T = (-C_{-1}\eta^{-1},0,\dots,0,C_{-1}\eta^{-1})^T + Sv + \mathsf{O}(p) \text{ for some } v.$$

Let $B = B_\alpha$ be the $(\alpha+1)\times(\alpha+1)$ matrix defined in lemma 8. That lemma implies that B encodes precisely the row operations that transform S into a matrix with zeros outside the rows indexed $1,\dots,\beta$ and such that

$$(BS)_{w,j} = p^{-[j=0]}\binom{s+\beta(p-1)-\alpha+j}{w(p-1)+j}$$

when $w \in \{1,\dots,\beta\}$. Moreover,

$$B_{i,w} = [(i,w)=(0,0)] + \textstyle\sum_{l=1}^{\alpha}(-1)^i\binom{l}{i}\binom{l-1}{l-w} + \mathsf{O}(p).$$

By using this formula we can compute that

$$B(-1,0,\dots,0,1)^T = \left(0,-\binom{\alpha}{1},\dots,(-1)^\alpha\binom{\alpha}{\alpha}\right)^T + \mathsf{O}(p),$$

and therefore if $\overline{R}$ is the $\alpha\times\alpha$ matrix over $\mathbb{F}_p$ obtained from $\overline{BN}$ by replacing the rows indexed $1,\dots,\beta$ with the corresponding rows of $\overline{BS}$ and then discarding the zeroth row and the zeroth column, the condition that needs to be satisfied is equivalent to the claim that

$$\overline{R}(C_1,\dots,C_\alpha)^T = \left(-(1-[1\leqslant\beta])\binom{\alpha}{1},\dots,(-1)^\alpha(1-[\alpha\leqslant\beta])\binom{\alpha}{\alpha}\right)^T. \tag{3}$$

The matrix $\overline{R}$ is the lower right $\alpha\times\alpha$ submatrix of the matrix $\overline{Q}$ defined in the proof of proposition 19, where we compute that

$$\overline{R}_{i-1,j-1} = \binom{s-\alpha-\beta+j}{j-i}\cdot\begin{cases}\binom{\beta}{i} & \text{if } i\in\{1,\dots,\beta\},\\ -\binom{\beta}{i}^{\partial} & \text{otherwise,}\end{cases}$$

for $i,j\in\{1,\dots,\alpha\}$. We have

$$\begin{aligned}\textstyle\sum_{j=1}^{\alpha}&(-1)^{\alpha+\beta+j}\alpha\binom{\alpha-1}{\beta}\binom{s-\alpha+1}{\alpha-j}\binom{s-\alpha-\beta+j}{j-i}\\ &= \textstyle\sum_j(-1)^{\alpha+\beta+i}\alpha\binom{\alpha-1}{\beta}\binom{s-\alpha+1}{\alpha-j}\binom{\alpha+\beta-s-i-1}{j-i}\\ &= (-1)^{\alpha+\beta+i}\alpha\binom{\alpha-1}{\beta}\binom{\beta-i}{\alpha-i}.\end{aligned}$$

If $i\in\{1,\dots,\beta\}$ then the last expression is zero, and if $i\in\{\beta+1,\dots,\alpha\}$ then it is

$$(-1)^\beta\alpha\binom{\alpha-1}{\beta}\binom{\alpha-\beta-1}{i-\beta-1} = (-1)^\beta i\binom{i-1}{\beta}\binom{\alpha}{i} = (-1)^i\binom{\alpha}{i}\cdot\left(-\binom{\beta}{i}^{\partial}\right)^{-1},$$

which implies (3). Consequently we can apply part (2) of corollary 17 with $v=t$ and conclude that $\operatorname{ind}_{KZ}^{G}\operatorname{sub}(\alpha)$ is trivial modulo $\mathscr{I}_a$. ∎

6. Proof of theorem 4

The proof of theorem 4 is very similar to the proof of theorem 3. The major difference is that we apply corollary 18 instead of corollary 17 since $v_p(a)$ is an integer, which means that $\overline{\Theta}_{k,a}$ is reducible in some cases.

We make the same assumptions as in section 5. We assume that

$$r = s + \beta(p-1) + u_0p^t + \mathsf{O}\left(p^{t+1}\right)$$

for some $\beta \in \{0, \dots, p-1\}$ and $u_0 \in \mathbb{Z}_p^\times$ and $t \in \mathbb{Z}_{>0}$, that $\eta = u_0 p^t$, that $v_p(a) = \nu - 1$ for some $\nu \in \{1, \dots, \frac{p-1}{2}\}$, that $s \in \{2\nu, \dots, p-2\}$, and that $k > p^{100}$ (and consequently $r > p^{99}$).

Let $\mu = \lambda$ if $k \in \mathscr{R}_0^{s,\nu}$ and $\mu = \lambda_{\beta,t}$ if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ (in the notation of theorem 2). Let $\gamma = \nu - 1$ if $k \in \mathscr{R}_0^{s,\nu}$ and $\gamma = \max\{\nu - t - 1, \beta\}$ if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$. Let us first show the equivalence between theorem 4 and the following proposition.

Proposition 23. *Let either $k \in \mathscr{R}_0^{s,\nu}$, or $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ and $t < \nu - \beta - 1$.*

(1) $(T - \mu^{-1})(\mathrm{ind}_{KZ}^G \mathrm{quot}(\gamma - 1))$ *is trivial modulo $\mathscr{I}_a$.*

(2) $(T - \mu)(\mathrm{ind}_{KZ}^G \mathrm{sub}(\gamma))$ *is trivial modulo $\mathscr{I}_a$.*

(3) $\mathrm{ind}_{KZ}^G \mathrm{sub}(\gamma - 1)$ *is trivial modulo $\mathscr{I}_a$.*

(4) $\mathrm{ind}_{KZ}^G \mathrm{quot}(\gamma)$ *is trivial modulo $\mathscr{I}_a$.*

Proof that theorem 4 is equivalent to proposition 23. First let us assume that proposition 23 is true. In the setting of theorem 3, propositions 19, 20, 21, and 22 show that if $k \in \mathscr{R}_0^{s,\nu}$ then $\overline{\Theta}_{k,a}$ is a quotient of $\mathrm{ind}_{KZ}^G \mathrm{quot}(\nu - 1)$, and if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ then $\overline{\Theta}_{k,a}$ is a quotient of $\mathrm{ind}_{KZ}^G \mathrm{quot}(\max\{\nu - t - 1, \beta\})$. Their proofs are based on corollary 17. They amount to considering the element of $\mathrm{im}(T - a)$ coming from equation 1 in lemma 16, and noting that the term with dominant valuation is either H or

$$(\vartheta' + C_{-1}) \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^\alpha x^{p-1} y^{r-\alpha(p+1)-p+1} + C_{-1} \bullet_{KZ,\overline{\mathbb{Q}}_p} \theta^n x^{\alpha-n} y^{r-np-\alpha},$$

depending on how t compares to $v_p(a) - \alpha$. In the setting of theorem 4 we can apply the analogous corollary 18 to conclude that any infinite-dimensional factor of $\overline{\Theta}_{k,a}$ must be a quotient of one of

$$\mathrm{ind}_{KZ}^G \mathrm{sub}(\gamma - 1),\ \mathrm{ind}_{KZ}^G \mathrm{quot}(\gamma - 1),\ \mathrm{ind}_{KZ}^G \mathrm{sub}(\gamma),\ \mathrm{ind}_{KZ}^G \mathrm{quot}(\gamma),$$

where for convenience we define $\mathrm{sub}(-1)$ and $\mathrm{quot}(-1)$ to be the trivial representation. The key reason why the proofs of propositions 19 and 20 copy verbatim to prove this is that outside of these subquotients the valuations t and $v_p(a) - \alpha$ never match, so again exactly one of the two aforementioned terms is dominant. The only subtlety when copying the proofs of propositions 19 and 20 is that we do not know whether $\mathscr{I}_a$ contains $1 \bullet_{KZ,\overline{\mathbb{Q}}_p} x^{\nu-1} y^{r-\nu+1}$. This ultimately does not present a problem since when working with $\widehat{N}_{\nu-1}$ we always assume that $C_0 = 0$. As the proofs of propositions 19 and 20 work here nearly without modification except for replacing corollary 17 with 18, we omit the full details of the arguments. Proposition 23 then implies that any infinite-dimensional factor of $\overline{\Theta}_{k,a}$ must be a quotient of one of

$$\mathrm{ind}_{KZ}^G \mathrm{quot}(\gamma - 1)/(T - \mu^{-1}),\ \mathrm{ind}_{KZ}^G \mathrm{sub}(\gamma)/(T - \mu),$$

and together with theorem 2 in [Ars21] they completely determine $\overline{\Theta}_{k,a}$.

The converse, that theorem 3 implies proposition 23, is clear since theorem 3 completely determines $\overline{\Theta}_{k,a}$. ■

Proof of proposition 23.

(1) Suppose first that $k \in \mathscr{R}_0^{s,\nu}$. Let $c = (c_1, \dots, c_\alpha)$ be as in lemma 10, and let us make the substitutions $X = r - \alpha$ and $Y = s - \alpha$. We apply part (6) of corollary 18 with $v = 1$. We choose $C_{-1} = 0$ and $C_0 = 1$ and $C_j = \frac{c_j p}{(s-\alpha)_\alpha}$ for $j \in \{1, \dots, \alpha\}$. In the proof of proposition 19 we show that for these constants we have

$$\begin{aligned} \vartheta' &= \tfrac{(s-r)_{\nu-1}}{(s-\nu+2)_{\nu-1}} p + \mathsf{O}\big(p^2\big), \\ 1 &\leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha \leqslant w < 2\nu - \alpha, \\ 1 &< v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha. \end{aligned}$$

Moreover, since $C_1, \dots, C_\alpha = \mathsf{O}(p)$ we have $\check{C} = \mathsf{O}(p)$, and

$$\mu = \tfrac{(s-\nu+2)_{\nu-1} a}{(s-r)_{\nu-1} p^\nu} = \tfrac{a}{C_0^{-1} \vartheta' p^{\nu-2}}.$$

Therefore the conditions needed to apply part (6) of corollary 18 (i.e. the three conditions

$$\begin{aligned} v &\leqslant v_p(\vartheta_\alpha(D_\bullet)), \\ v' := \min\{v_p(a) - \alpha, v\} &\leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha < w < 2\nu - \alpha, \\ v' &< v_p(\vartheta_w(D_\bullet)) \text{ for } 0 \leqslant w < \alpha, \end{aligned}$$

in addition to the two extra conditions on $\check{C}$ and C_0 in part (6) of the corollary) are satisfied and we can conclude that

$$(T - \mu^{-1})(\operatorname{ind}_{KZ}^G \operatorname{quot}(\gamma - 1))$$

is trivial modulo $\mathscr{I}_a$. If $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ and $t < \nu - \beta - 1$ then the argument is similar: we choose $v = t + 1$ and the same constants $\{C_j\}$ as in the third bullet point (if $\beta = 0$) or the fourth bullet point (if $\beta \in \{1, \dots, \gamma - 1\}$) of the proof of proposition 19. In the former case

$$\vartheta' = \tfrac{(s-r)_{\alpha+1}}{(s-\alpha)_{\alpha+1}} p + \mathsf{O}\big(p^{t+2}\big),$$

and in the latter case

$$\vartheta' = \epsilon Q_{0,0} p + \mathsf{O}\big(p^{t+2}\big).$$

In both cases

$$\mu = \tfrac{a}{C_0^{-1} \vartheta' p^{\nu-t-2}}$$

and the conditions needed to apply part (6) of corollary 18 are satisfied, so again we can conclude that

$$(T - \mu^{-1})(\operatorname{ind}_{KZ}^G \operatorname{quot}(\gamma - 1))$$

is trivial modulo $\mathscr{I}_a$.

(2) Suppose first that $k \in \mathscr{R}_0^{s,\nu}$. We use the constants

$$C_j = \begin{cases} (-1)^{\nu-1} \binom{s-r}{\nu-1} & \text{if } j = -1, \\ 0 & \text{if } j = 0, \\ (-1)^{\nu-j-1} \binom{s-\nu+2}{\nu-j-1} & \text{if } j \in \{1, \dots, \alpha\}. \end{cases}$$

We can show just as in the proof of proposition 21 that these constants satisfy all of the conditions needed to apply part (7) of corollary 18 with $v = 0$. Moreover, we have

$$\begin{aligned}\check{C} &= \textstyle\sum_{j=1}^{\nu-1}(-1)^{\nu-j-1}\binom{s-\nu+2}{\nu-j-1}\binom{r-\nu+j+1}{j} - (-1)^{\nu-1}\binom{s-r}{\nu-1} + \mathsf{O}(p)\\ &= (-1)^{\nu-1}\textstyle\sum_{j=1}^{\nu-1}\binom{s-\nu+2}{\nu-j-1}\binom{\nu-r-2}{j} - (-1)^{\nu-1}\binom{s-r}{\nu-1} + \mathsf{O}(p)\\ &= (-1)^{\nu}\binom{s-\nu+2}{\nu-1} + \mathsf{O}(p).\end{aligned}$$

Thus

$$-\frac{\check{C}ap^{1-\nu}}{C_{-1}} = \frac{\binom{s-\nu+2}{\nu-1}a}{\binom{s-r}{\nu-1}p^{\nu-1}} = \mu,$$

so part (7) of corollary 18 implies that

$$(T-\mu)(\operatorname{ind}_{KZ}^{G}\operatorname{sub}(\gamma))$$

is trivial modulo $\mathscr{I}_a$. If $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ and $t < \nu-\beta-1$ then the argument is similar: we choose $v = t$ and the constants

$$C_j = \begin{cases} \eta & \text{if } j=-1,\\ 0 & \text{if } j=0,\\ (-1)^{\gamma+\beta+j}\gamma\binom{\gamma-1}{\beta}\binom{s-\gamma+1}{\gamma-j} & \text{if } j\in\{1,\ldots,\gamma\},\end{cases}$$

as in the proof of proposition 22. Again all of the conditions needed to apply part (7) of corollary 18 with $v = t$ are satisfied and

$$\begin{aligned}\check{C} &= (-1)^{\beta+\gamma}\gamma\binom{\gamma-1}{\beta}\textstyle\sum_{j=1}^{\gamma}(-1)^{j}\binom{s-\gamma+1}{\gamma-j}\binom{r-\gamma+j}{j} + \mathsf{O}(p)\\ &= (-1)^{\beta+\gamma}\gamma\binom{\gamma-1}{\beta}\textstyle\sum_{j=1}^{\gamma}\binom{s-\gamma+1}{\gamma-j}\binom{\gamma-r-1}{j} + \mathsf{O}(p)\\ &= (-1)^{\beta+\gamma}\gamma\binom{\gamma-1}{\beta}\left(\binom{s-r}{\gamma} - \binom{s-\gamma+1}{\gamma}\right) + \mathsf{O}(p)\\ &= (-1)^{\beta+\gamma+1}\gamma\binom{\gamma-1}{\beta}\binom{s-\gamma+1}{\gamma} + \mathsf{O}(p).\end{aligned}$$

The last equality follows from the fact that $\binom{s-r}{\gamma} = \mathsf{O}(p)$. Thus

$$-\frac{\check{C}a}{C_{-1}p^{\nu-t-1}} = \frac{(-1)^{\beta+\gamma}\gamma\binom{\gamma-1}{\beta}\binom{s-\gamma+1}{\gamma}a}{\epsilon p^{\nu-t-1}} = \mu,$$

so part (7) of corollary 18 implies that

$$(T-\mu)(\operatorname{ind}_{KZ}^{G}\operatorname{sub}(\gamma))$$

is trivial modulo $\mathscr{I}_a$.

(3) This is very similar to part (2) of this proposition: if $k \in \mathscr{R}_0^{s,\nu}$ then we use part (2) of corollary 18 just as in the proof of proposition 21, and if $k \in \mathscr{R}_{\beta,t}^{s,\nu}$ and $t < \nu-\beta-1$ then we use part (2) of corollary 18 just as in the proof of proposition 22. We omit the full details.

(4) We apply part (8) of corollary 18 with the same constants as in the proof of part (2) of this proposition—since $\check{C} \in \mathbb{Z}_p^{\times}$ all of the necessary conditions are satisfied and we can conclude that $\operatorname{ind}_{KZ}^{G}\operatorname{quot}(\gamma)$ is trivial modulo $\mathscr{I}_a$. ∎

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