LIMITING MEASURES OF SUPERSINGULARITIES

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Abstract. Let p be a prime number and let $k \geqslant 2$ be an integer. In this article we study the semi-simple reductions modulo p of two-dimensional irreducible crystalline p-adic Galois representations with Hodge-Tate weights 0 and $k-1$ and large slopes. Berger–Li–Zhu proved by using the theory of (φ, Γ)-modules that this reduction is constant when the slope is larger than $\lfloor \frac{k-2}{p-1} \rfloor$. Recently, Bergdall–Levin improved this bound to $\lfloor \frac{k-1}{p} \rfloor$ by using the theory of Kisin modules. In this article, under the extra assumptions $p > 3$ and $p+1 \nmid k-1$, we asymptotically improve this bound further to $\lfloor \frac{k-1}{p+1} \rfloor + \lfloor \log_p(k-1) \rfloor$, which is off from the predicted optimal bound $\approx \frac{k-1}{p+1}$ only by a factor of $\mathsf{O}\big(\log_p k\big)$ rather than by a factor that is linear in k. As a consequence we deduce a partial result towards a conjecture by Gouvêa: that the measures of supersingularities of level Np oldforms tend to the zero measure on the interval $(\frac{1}{p+1}, \frac{p}{p+1})$ when p is coprime to $6N$ and $\Gamma_0(N)$-regular. It is very likely that our methods extend to the cases $p \in \{2, 3\}$ and $p+1 \nmid k-1$ as well, and therefore can be adapted to eliminate the extra assumptions $p > 3$ and $p+1 \nmid k-1$.

1. Introduction

1.1. **Motivation.** Let p be a prime number and let $k \geqslant 2$ and N be positive integers such that N is coprime to p. In [Gou01], Gouvêa studied the "supersingularities" of weight k, level $\Gamma_0(Np)$ eigenforms. To be specific, if f is such an eigenform with slope v, we define the supersingularity of f as $\frac{v}{k-1}$. The supersingularities of weight k, level $\Gamma_0(Np)$ eigenforms belong to the interval $[0,1]$ and are symmetric under $\eta \leftrightarrow 1-\eta$, and we denote by μ_k their discrete probability measure, i.e. the probability measure on the interval $[0,1]$ we obtain by putting a point mass at each supersingularity. For $N=1$, his computations showed that the measure μ_k is supported on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$ most of the time, and the only exceptions that occurred in his computations were for the primes p belonging to the set

$$E_1 = \{59, 79, 2411, 3371, 15271, 64709, 187441, 27310421\}. \tag{1}$$

Even the exceptional supersingularities seemed to approach either $\frac{1}{p+1}$ or $\frac{p}{p+1}$ as $k \to \infty$, leading Gouvêa to the following conjecture.

Conjecture A (Gouvêa). *The sequence $(\mu_l)_{l \geqslant 2}$ converges to the uniform measure on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$.*

This is the p-adic version of an interesting twist on the Sato–Tate conjecture: while the Sato–Tate conjecture asks about the distribution of the (real) slopes of a fixed modular form for varying primes, here one is interested in the distributions of the (p-adic) slopes of varying modular forms for a fixed prime.

Each newform of level $\Gamma_0(N)$ maps onto a pair of oldforms f_1, f_2 of level $\Gamma_0(Np)$ via the maps defined on q-expansions that send $q \mapsto q$ and $q \mapsto q^p$. If the slope of f is α, then the slopes of f_1, f_2 are α and $k-1-\alpha$; moreover, every newform of level $\Gamma_0(Np)$ has slope $\frac{k-2}{2}$ (see section 1 of [Gou01]). This is why μ_k is symmetric under $\eta \leftrightarrow 1-\eta$, and it implies that finding the slopes of eigenforms of level $\Gamma_0(Np)$ is equivalent to finding the slopes of newforms of level $\Gamma_0(N)$.

In [Buz05], Buzzard introduced the notion of "$\Gamma_0(N)$-regularity" and made some concrete conjectures about the slopes of modular forms for $\Gamma_0(N)$-regular primes. This notion is related to Gouvêa's conjecture because all of the exceptional primes in the set E_1 are $\Gamma_0(1)$-irregular, which led to the prediction that μ_k is supported on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$ when the prime p is $\Gamma_0(N)$-regular. One avenue to attempt to prove this prediction is via (p-adic) Galois representations, because the notion of $\Gamma_0(N)$-regularity can be rephrased in terms of Galois representations when $p > 3$: a prime $p > 3$ is $\Gamma_0(N)$-regular if and only if, for all weights $l \geqslant 2$ and all $f \in S_l(\Gamma_0(N))$, the modulo p Galois representation associated with f is reducible. This also naturally leads to the more general prediction that all exceptions to Gouvêa's observation happen only for modular forms whose associated modulo p Galois representation is reducible.

1.2. **Galois representations.** The p-adic Galois representations $V_{k,a}$ were introduced by Colmez–Fontaine in [CF00], and we denote their semi-simple reductions modulo p by $\overline{V}_{k,a}$. They relate to the Galois representations associated with modular forms because the Galois representation $\overline{\rho}_f$ associated with an eigenform f of weight k, level $\Gamma_0(N)$, character χ, and U_p-eigenvalue a_p is

$$\rho_f = V_{k,a_p\sqrt{\chi}} \otimes \sqrt{\chi}. \tag{2}$$

Therefore, if $p > 3$ is $\Gamma_0(N)$-regular then the statement

$$\text{“}\overline{V}_{k,a} \text{ reducible} \implies v_p(a) \leqslant \alpha(k) \text{ for a function } \alpha : \mathbb{Z} \to \mathbb{Q}\text{”} \tag{3}$$

implies that μ_k is supported on $[0, \frac{\alpha(k)}{k-1}] \cup [1 - \frac{\alpha(k)}{k-1}, 1]$.

Berger–Li–Zhu proved in [BLZ04] by using the theory of (φ, Γ)-modules that the representations $\overline{V}_{k,a}$ are locally constant around the infinite slope $a = 0$, and they gave an explicit lower bound for the radius of the region of local constancy. To be more specific, they showed the following theorem.

Theorem 1 (Berger–Li–Zhu)**.** *If $v_p(a) > \lfloor \frac{k-2}{p-1} \rfloor$ then $\overline{V}_{k,a} \cong \overline{V}_{k,0}$.*

Since $\overline{V}_{k,0} \cong \mathrm{ind}(\omega_2^{k-1})$ is irreducible when k is even (see for example proposition 6.1.2 in [Bre03]), and there are no nontrivial eigenforms of odd weight, theorem 1 implies the following corollary.

Corollary 2. *If $p > 3$ is $\Gamma_0(N)$-regular then each of restrictions*

$$\left(\mu_l|_{(\frac{1}{p-1}, \frac{p-2}{p-1})}\right)_{l \geqslant 2} \tag{4}$$

is the zero measure on $(\frac{1}{p-1}, \frac{p-2}{p-1})$.

In light of Gouvêa's conjecture, the lower bound $\lfloor \frac{k-2}{p-1} \rfloor$ in theorem 1 is believed to be suboptimal. The optimal bound is believed to be closer to $\frac{k-1}{p+1}$. Recently, Bergdall–Levin improved the bound to $\lfloor \frac{k-1}{p} \rfloor$ by using the theory of Kisin modules ([BL]). To be more specific, they showed the following theorem.

Theorem 3 (Bergdall–Levin). *If $v_p(a) > \lfloor \frac{k-1}{p} \rfloor$ then $\overline{V}_{k,a} \cong \overline{V}_{k,0}$.*

A consequence of this theorem is the followning corollary, which is the analogue of corollary 2 with the interval $(\frac{1}{p-1}, \frac{p-2}{p-1})$ replaced by the interval $(\frac{1}{p}, \frac{p-1}{p})$.

Corollary 4. *If $p > 3$ is $\Gamma_0(N)$-regular then each of restrictions*

$$\Big(\mu_l|_{(\frac{1}{p-1}, \frac{p-2}{p-1})}\Big)_{l \geqslant 2} \tag{5}$$

is the zero measure on $(\frac{1}{p}, \frac{p-1}{p})$.

1.3. **Main results of this article.** For simplicity, we assume the extra condition $p > 3$. It is very likely that our methods extend to the case $p \in \{2, 3\}$ as well, and therefore can be adapted to eliminate the extra assumption $p > 3$.

Theorem M. *If $p > 3$ and $p+1 \nmid k-1$ and*

$$v_p(a) > \lfloor \tfrac{k-1}{p+1} \rfloor + \lfloor \log_p(k-1) \rfloor \tag{6}$$

then $\overline{V}_{k,a} \cong \overline{V}_{k,0}$.

In other words, under the extra conditions $p > 3$ and $p+1 \nmid k-1$, we asymptotically improve the bounds in theorems 1 and 3 to

$$\lfloor \tfrac{k-1}{p+1} \rfloor + \lfloor \log_p(k-1) \rfloor = \lfloor \tfrac{k-1}{p+1} \rfloor + \mathsf{O}\big(\log_p k\big), \tag{7}$$

which is off from the the predicted optimal bound $\approx \frac{k-1}{p+1}$ only by an additive factor of $\mathsf{O}\big(\log_p k\big)$ rather than by an additive factor that is linear in k.

The case $p+1 \nmid k-1$ is of local interest, but if $p > 3$ and $p+1 \mid k-1$ then $2 \nmid k$, and there are no nontrivial eigenforms of odd weight and level $\Gamma_0(N)$, so theorem M is enough to imply the following global result.

Corollary C. *If $p > 3$ is $\Gamma_0(N)$-regular then the sequence of restrictions*

$$\Big(\mu_l|_{(\frac{1}{p+1}, \frac{p}{p+1})}\Big)_{l \geqslant 2} \tag{8}$$

converges to the zero measure on $(\frac{1}{p+1}, \frac{p}{p+1})$.

With conjecture A in mind, the interval $(\frac{1}{p+1}, \frac{p}{p+1})$ is optimal here.

The proof of theorem M is based on the local Langlands correspondence, and there are several known results that we use. We rely on Berger–Li–Zhu's theorem for the region already covered by it. Outside that region we rely on a theorem by Chenevier–Colmez on the continuity of a family of trianguline representations that $\overline{V}_{k,a}$ belongs to. Counter-intuitively, the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated with $\overline{V}_{k,a}$ itself via the local Langlands correspondence is too unwieldy, and the application of Chenevier–Colmez's theorem is crucial as it gives us a host of nearby representations to work with instead. We use the approximation results from [Ars21a] to relate $\overline{V}_{k,a}$ to these nearby representations, and thus get the best of both worlds: the simplicity of the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated with $\overline{V}_{k,a}$ (a result of the "weight" parameter k of $\overline{V}_{k,a}$ being small relative to the slope), and the structural richness of the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations associated with the nearby representations (a result of the weight parameters of these nearby representations being large relative to the slope).

1.4. **Organization.** In section 2 ("Definitions") we introduce all of the necessary definitions, and we also recall the p-adic and modulo p local Langlands correspondences in the form that is the most convenient for this article. In section 3 ("General results about $\mathrm{GL}_2(\mathbb{Q}_p)$-representations") we prove some general results about the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations associated with Galois representations like $\overline{V}_{k,a}$ via the local Langlands correspondence. Section 4 ("Proofs") is the most difficult one: it contains the bulk of the ideas that make up the proof of theorem M. The proof contains a rather significant amount of rather difficult combinatorics. In order to improve readability, we defer many of the combinatorial results to section 5 ("Combinatorics"); the reader is encouraged to treat that section as a black box on first reading. As we use several results from [Ars21a], in order to be economical with space we refer to that article for some general and combinatorial results in sections 3 and 5.

2. Definitions

2.1. **Notation.** For the remainder of this article we assume that p is a prime number and k and N are positive integers. For simplicity, we assume that $p > 3$, though it is very likely that this assumption can be removed by suitably adjusting our combinatorial results. We also assume that $N \in \mathbb{Z}\backslash p\mathbb{Z}$ is coprime to p. We assume that $a \in \overline{\mathbb{Z}}_p$ is such that $v_p(a) > \lfloor \frac{k-1}{p+1} \rfloor \geqslant 0$. The modulo p Galois representation $\overline{V}_{k,a}$ is defined in section 2 of [Ars21a] and subsection 1.1 of [BLZ04], so we do not reproduce its definition here. In addition to $\overline{V}_{k,a}$ we also consider "nearby" representations $\overline{V}_{k',a'}$, where $k' \approx k$ (i.e. k' is very close to k in the p-adic norm) and $a' \approx a$. To define these precisely, we note that a local constancy theorem by Chenevier–Colmez (see proposition 4.13 in [Col08] and proposition 3.9 in [Che13] for the results, and section 4 of [Ars21b] for further clarification on how they are applied in this setting) implies that

$$\overline{V}_{k,a} \cong \overline{V}_{k+(p-1)p^m,\mathfrak{a}_m} \tag{9}$$

for some integer $M = M(k,a) > 0$, all integers $m \geqslant M$, and a certain sequence $(\mathfrak{a}_m)_{m\geqslant M}$ consisting of elements of $\overline{\mathbb{Z}}_p$ such that $v_p(\mathfrak{a}_m) = v_p(a)$ for all $m \geqslant M$. For the remainder of this article we fix an integer $M = M(k,a)$ and a sequence $(\mathfrak{a}_m)_{m\geqslant M}$ with this property. We define

$$\begin{aligned} r &= k-2, \\ \varrho &= \lfloor \tfrac{k-1}{p+1} \rfloor = \lfloor \tfrac{r+1}{p+1} \rfloor, \\ \mathcal{E} &= \lfloor \log_p(k-1) \rfloor. \end{aligned} \tag{10}$$

We want to choose integers ϵ, δ, η that satisfy

$$v_p(\epsilon) \gg v_p(\delta) \gg v_p(\eta), \tag{11}$$

and an element $\mathfrak{a} \in \overline{\mathbb{Z}}_p$ which has the same valuation as a such that $\overline{V}_{k+\epsilon,\mathfrak{a}} \cong \overline{V}_{k,a}$. That way we can compute the representation $\overline{V}_{k,a}$ by computing the isomorphic "nearby" representation $\overline{V}_{k+\epsilon,\mathfrak{a}}$. The following explicitly chosen parameters (which depend on k and a) work:

$$\begin{aligned}\eta &= p^{100kM\lceil v_p(a)\rceil} \in \mathbb{Z},\\ \delta &= \eta^{100\eta} \in \mathbb{Z},\\ \epsilon &= (p-1)\delta^{100\delta} = (p-1)p^{1000000kM\lceil v_p(a)\rceil\eta\delta} \in \mathbb{Z},\\ t &= r+\epsilon,\\ \mathfrak{a} &= \mathfrak{a}_{1000000kM\lceil v_p(a)\rceil\eta\delta} \in \overline{\mathbb{Z}}_p, \text{ satisfying } v_p(\mathfrak{a}) = v_p(a) > \varrho = \lfloor \tfrac{k-1}{p+1} \rfloor,\\ \overline{V} &= \overline{V}_{t+2,\mathfrak{a}} = \overline{V}_{k+\epsilon,\mathfrak{a}} \cong \overline{V}_{k,a} \text{ by equation (9).}\end{aligned} \tag{12}$$

In light of theorem 1, theorem M is true if $v_p(a) > \lfloor \frac{k-2}{p-1} \rfloor$. Therefore, for the remainder of this article we assume additionally that

$$\lfloor \tfrac{k-2}{p-1} \rfloor \geqslant v_p(a) > \lfloor \tfrac{k-1}{p+1} \rfloor + \mathcal{E}. \tag{13}$$

In particular, we note that $a^2 \neq 4p^{k-1}$ and $a \neq \pm(1+p^{-1})p^{k/2}$ and $\mathfrak{a}^2 \neq 4p^{k+\epsilon-1}$ and $\mathfrak{a} \neq \pm(1+p^{-1})p^{(k+\epsilon)/2}$; these are eigenvalues that could potentially cause problems with the local Langlands correspondence.

2.2. **Local Langlands.** Let B be the Borel subgroup of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ consisting of the upper triangular elements, let $K = \mathrm{GL}_2(\mathbb{Z}_p) \subset G = \mathrm{GL}_2(\mathbb{Q}_p)$, and let Z be the center of G. Let μ_x be the unramified character of the Weil group that sends the geometric Frobenius to x, and let $|\ | : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times \hookrightarrow \overline{\mathbb{Q}}_p^\times$ be the p-adic norm. Let W be a finite-dimensional locally algebraic representation of the closed subgroup KZ of G. We define the compact induction of W by

$$\operatorname{ind}^G W := \big\{\text{locally algebraic } G \to W \;\big|\; f(hg) = hf(g) \text{ for all } h \in KZ \;\&\; \operatorname{supp} f \text{ is compact in } KZ\backslash G\big\}. \tag{14}$$

Suppose that W is over the field $\mathbb{F} \in \{\overline{\mathbb{Q}}_p, \overline{\mathbb{F}}_p\}$. For elements $g \in G$ and $w \in W$ we write $g \bullet_{H,\mathbb{F}} w$ for the unique element of $\operatorname{ind}^G W$ that is supported on KZg^{-1} and maps g^{-1} to w. Every element of $\operatorname{ind}^G W$ can be written as a finite linear combination of functions of the type $g \bullet_{H,\mathbb{F}} w$, and

$$g_1(g_2 \bullet_{H,\mathbb{F}} (hw)) = (g_1 g_2 h) \bullet_{H,\mathbb{F}} w. \tag{15}$$

For $l \geqslant 0$ we define

$$\widetilde{\Sigma}_l = \underline{\mathrm{Sym}}^l(\overline{\mathbb{Q}}_p^2) := \mathrm{Sym}^l(\overline{\mathbb{Q}}_p^2) \otimes |\det|^{l/2}, \tag{16}$$

and we define Σ_l as the reduction of $\underline{\mathrm{Sym}}^l(\overline{\mathbb{Z}}_p^2)$ modulo the maximal ideal $\mathfrak{m}$ of $\overline{\mathbb{Z}}_p$. For $h \in \mathbb{Z}$ we define

$$\sigma_h := \underline{\mathrm{Sym}}^h(\overline{\mathbb{F}}_p^2). \tag{17}$$

As in section 3 of [Ars21a], we note that we can view these as G-modules of homogeneous polynomials in two variables.

Let us also define the Hecke operator $T \in \mathrm{End}_G(\operatorname{ind}^G \widetilde{\Sigma}_t)$ corresponding to the double coset of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This operator satisfies the explicit formula

$$T(\gamma \bullet_{\overline{\mathbb{Q}}_p} v) = \textstyle\sum_{\mu \in \mathbb{F}_p} \gamma\left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} \left(\left(\begin{smallmatrix} 1 & -[\mu] \\ 0 & p \end{smallmatrix}\right) \cdot v\right) + \gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} \left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot v\right), \tag{18}$$

where $[\xi]$ is the Teichmüller lift of $\xi \in \mathbb{F}_p$ to $\mathbb{Z}_p$. We define

$$\begin{aligned}\Pi_{t+2,\mathfrak{a}} &= \operatorname{ind}^G \widetilde{\Sigma}_t/(T-a),\\ \Theta_{t+2,\mathfrak{a}} &= \operatorname{im}\left(\operatorname{ind}^G(\underline{\operatorname{Sym}}^t(\overline{\mathbb{Z}}_p^2)) \longrightarrow \Pi_{t+2,\mathfrak{a}}\right),\\ \overline{\Theta}_{t+2,\mathfrak{a}} &= \Theta_{t+2,\mathfrak{a}} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p. \end{aligned} \tag{19}$$

In particular, $\overline{\Theta}_{t+2,\mathfrak{a}}$ is a quotient of $\operatorname{ind}^G \Sigma_t$, and we define $\mathscr{I}$ to be the ideal such that

$$\overline{\Theta}_{t+2,\mathfrak{a}} \cong \operatorname{ind}^G \Sigma_t/\mathscr{I}. \tag{20}$$

The ideal $\mathscr{I}$ contains the reduction modulo p of any integral element in the image of $T-\mathfrak{a}$. For $j \in \{0,\dots,p-1\}$, $\lambda \in \overline{\mathbb{F}}_p$, and a character $\psi : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$, we write

$$\pi(j,\lambda,\psi) := (\operatorname{ind}^G \sigma_t/(T_\sigma - \lambda)) \otimes \psi, \tag{21}$$

where $T_\sigma \in \operatorname{End}_G(\operatorname{ind}^G \sigma_j)$ is the Hecke operator corresponding to the double coset of $\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)$. We let ω be the modulo p reduction of the cyclotomic character, $\operatorname{ind}(\omega_2^{j+1})$ be the unique irreducible representation whose determinant is ω^{j+1} and that is equal to $\omega_2^{j+1} \oplus \omega_2^{p(j+1)}$ on inertia, $\overline{h} \in \{1,\dots,p-1\}$ and $\underline{h} \in \{0,\dots,p-2\}$ be the numbers in the corresponding sets that are congruent to h modulo $p-1$. The following theorem is the main result of [Ber10] and says that the modulo p Langlands correspondence is compatible with the p-adic local Langlands correspondence.

Theorem 5. *There are $j \in \{0,\dots,p-1\}$ and $\psi : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$ such that either*

$$\overline{\Theta}_{k,a} \cong \pi(j,0,\psi) \tag{22}$$

or

$$\overline{\Theta}_{k,a}^{\mathrm{ss}} \cong \left(\pi(j,\lambda,\psi) \oplus \pi(\underline{p-3-j},\lambda^{-1},\omega^{j+1}\psi)\right)^{\mathrm{ss}} \tag{23}$$

for some $\lambda \in \overline{\mathbb{F}}_p$. In the former case we have

$$\overline{V}_{k,a} \cong \operatorname{ind}(\omega_2^{j+1}) \otimes \psi, \tag{24}$$

and in the latter case we have

$$\overline{V}_{k,a} \cong \left(\mu_\lambda \omega^{j+1} \oplus \mu_{\lambda^{-1}}\right) \otimes \psi. \tag{25}$$

Let $\overline{\Theta} = \overline{\Theta}_{t+2,\mathfrak{a}}^{\mathrm{ss}}$. Proposition 4.1.4 in [BLZ04] implies that

$$\overline{V}_{k,0} \cong \begin{cases} \operatorname{ind}(\omega_2^{k-1}) & \text{if } p+1 \nmid k-1,\\ \left(\mu_{\sqrt{-1}} \oplus \mu_{-\sqrt{-1}}\right) \otimes \omega^{(k-1)/(p+1)} & \text{if } p+1 \mid k-1. \end{cases} \tag{26}$$

Therefore, in order to prove theorem M, we want to show that

$$\overline{V} = \overline{V}_{t+2,\mathfrak{a}} \cong \operatorname{ind}(\omega_2^{k-1}), \tag{27}$$

(since $p+1 \nmid k-1$). So theorem 5 implies that theorem M can be rewritten in the following equivalent form.

Theorem M'. *Recall that $p > 3$ and $p+1 \nmid k-1$ and $v_p(\mathfrak{a}) > \lfloor \frac{k-1}{p+1} \rfloor + \mathcal{E}$. We have*

$$\overline{\Theta} \cong \pi(\overline{r-2\varrho}, 0, \omega^\varrho). \tag{28}$$

So the goal of this article is to prove theorem M'.

2.3. **More notation.** Let $\nu = \lfloor v_p(\mathfrak{a}) \rfloor + 1$. We cite section 4 of [Ars21a] for the definitions of $\mathsf{O}(\alpha)$, and (for $h \in \mathbb{Z}$) I_h, and $\theta = xy^p - x^p y$, and the evaluation $[P]$ of a boolean P, which we do not reproduce here. We also recall that there is a filtration

$$\overline{\Theta} = \overline{\Theta}_0 \supset \overline{\Theta}_1 \supset \cdots \supset \overline{\Theta}_\alpha \supset \cdots \supset \overline{\Theta}_\nu = 0 \tag{29}$$

whose αth subquotient (for $\alpha \in \{0, \ldots, \nu - 1\}$) is a subquotient of

$$\widehat{N}_\alpha = \operatorname{ind}^G \left(\overline{\theta}^\alpha \Sigma_{t-\alpha(p+1)} / \overline{\theta}^{\alpha+1} \Sigma_{t-(\alpha+1)(p+1)} \right) \cong \operatorname{ind}^G I_{t-2\alpha}(\alpha) \cong \operatorname{ind}^G I_{r-2\alpha}(\alpha). \tag{30}$$

To be specific, if an element of

$$\operatorname{ind}^G \left(\overline{\theta}^\alpha \Sigma_{t-\alpha(p+1)} / \overline{\theta}^{\alpha+1} \Sigma_{t-(\alpha+1)(p+1)} \right) \tag{31}$$

is represented by an element of $\mathscr{I} \subset \operatorname{ind}^G \Sigma_t$, then that element is trivial in the subquotient $\overline{\Theta}_\alpha / \overline{\Theta}_{\alpha+1}$. Finally, we define (for $\alpha \in \{0, \ldots, \nu - 1\}$)

$$\begin{aligned} \operatorname{sub}(\alpha) &= \sigma_{\overline{r-2\alpha}}(\alpha) \subset N_\alpha, \\ \operatorname{quot}(\alpha) &= N_\alpha / \sigma_{\overline{r-2\alpha}}(\alpha) \cong \sigma_{\underline{2\alpha-r}}(r-\alpha), \end{aligned} \tag{32}$$

similarly as in section 4 of [Ars21a], and we denote by $T_{\mathrm{q},\alpha}, T_{\mathrm{s},\alpha}$ the Hecke operators corresponding to the double coset of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on the modules $\operatorname{ind}^G \operatorname{quot}(\alpha), \operatorname{ind}^G \operatorname{sub}(\alpha)$, respectively. For $\alpha \in \{0, \ldots, \delta\}$ we define

$$\begin{aligned} h_\alpha &= x^\alpha y^{t-\alpha} - x^{\alpha+\delta} y^{t-\alpha-\delta} \in \widetilde{\Sigma}_t, \\ h_\alpha^* &= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) h_\alpha = x^{t-\alpha} y^\alpha - x^{t-\alpha-\delta} y^{\alpha+\delta} \in \widetilde{\Sigma}_t. \end{aligned} \tag{33}$$

For $\alpha, \beta, R \geqslant 0$ we define $\Lambda_R(\alpha, \beta)$ by

$$\textstyle\sum_{\beta=\alpha-R}^{\alpha} \Lambda_R(\alpha, \beta) \binom{(p-1)X+\alpha}{\alpha-\beta} = \binom{R-X}{R} = (-1)^R \binom{X-1}{R} \in \mathbb{Q}_p[X]. \tag{34}$$

Note that both sides of equation (34) are polynomials in X over $\mathbb{Q}_p$ of degree R.

3. General results about $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

Lemma 6. *If $\alpha \in \{0, \ldots, \eta\}$ then*

$$\mathfrak{a} \bullet_{\overline{\mathbb{Q}}_p} h_\alpha \equiv_{\mathfrak{I}} p^\alpha \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} x^\alpha y^{t-\alpha} + \mathsf{O}\big(p^{2\eta}\big). \tag{35}$$

If $\alpha \in \{0, \ldots, \eta\}$, $\beta \in \{\alpha, \ldots, \eta\}$, and $(C_l)_{l \in \mathbb{Z}}$ is a family of elements of $\mathbb{Z}_p$ then

$$\begin{aligned} &\textstyle\sum_i \left(\sum_{l=\alpha-\beta}^{\alpha} C_l \binom{t-\alpha+l}{i(p-1)+l} \right) \bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha} \\ &\qquad \equiv_{\mathfrak{I}} \tfrac{\mathfrak{a} p^{-\alpha}}{p-1} \textstyle\sum_{l=\alpha-\beta}^{\alpha} C_l p^l \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-l} \left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h_{\alpha-l} + \mathsf{O}(p^\eta). \end{aligned} \tag{36}$$

Proof. We have

$$\begin{aligned} \mathfrak{a} \bullet_{\overline{\mathbb{Q}}_p} h_\alpha &\equiv_{\mathfrak{I}} T \left(1 \bullet_{\overline{\mathbb{Q}}_p} h_\alpha\right) \\ &\equiv_{\mathfrak{I}} \textstyle\sum_{\mu \in \mathbb{F}_p} \left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} A_\mu + \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} A, \end{aligned} \tag{37}$$

where, due to the explicit equation for T (equation (18)),

$$\begin{aligned} A_\mu &= x^\alpha(-[\mu]x+py)^{t-\alpha} - x^{\alpha+\delta}(-[\mu]x+py)^{t-\alpha-\delta} \\ &= \textstyle\sum_{\xi\geqslant 0}(-[\mu])^{\overline{t-\alpha-\xi}}\left(\binom{t-\alpha}{\xi}-\binom{t-\alpha-\delta}{\xi}\right)p^\xi x^{t-\xi}y^\xi = \mathsf{O}\big(\delta p^{-2\eta}+p^{2\eta}\big) = \mathsf{O}\big(p^{2\eta}\big), \end{aligned} \tag{38}$$

and

$$A = p^\alpha x^\alpha y^{t-\alpha} + \mathsf{O}\big(p^{\alpha+\delta}\big) = p^\alpha x^\alpha y^{t-\alpha} + \mathsf{O}\big(p^{2\eta}\big). \tag{39}$$

Equations (37), (38), and (39) imply equation (35). Equation (35) implies that

$$\begin{aligned} &\tfrac{\mathfrak{a}p^{-\alpha}}{p-1}\textstyle\sum_{l=\alpha-\beta}^{\alpha} C_l p^l \sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix} p & [\mu] \\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} h_{\alpha-l} \\ &\quad\equiv_{\mathfrak{I}} \tfrac{1}{p-1}\textstyle\sum_{l=\alpha-\beta}^{\alpha} C_l \sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix} 1 & [\mu] \\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{\alpha-l}y^{t-\alpha+l} + \mathsf{O}\big(p^{2\eta-\beta}\big) \\ &\quad\equiv_{\mathfrak{I}} \textstyle\sum_{l=\alpha-\beta}^{\alpha} C_l \sum_i \binom{t-\alpha+l}{i(p-1)+l}\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha} + \mathsf{O}(p^\eta), \end{aligned} \tag{40}$$

which implies equation (36). ∎

Lemma 7. *Let $\alpha \in \{0,\dots,\eta\}$ and $v\in\mathbb{Q}$ and the family $(D_i)_{i\in\mathbb{Z}}$ of elements of $\mathbb{Z}_p$ be such that*

$$\begin{aligned} &D_i = 0 \text{ for } i \notin [\tfrac{-\alpha}{p-1}, \tfrac{t-\alpha}{p-1}], \\ &v \leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha\leqslant w\leqslant 2\eta, \\ &v < v_p(\vartheta_w(D_\bullet)) \text{ for } 0\leqslant w<\alpha. \end{aligned} \tag{41}$$

For $j\in\mathbb{Z}$, let

$$\Delta_j = (-1)^{j-\eta}(1-p)^{-\alpha}\tbinom{\alpha}{j-\eta}\vartheta_\alpha(D_\bullet), \tag{42}$$

so that $(\Delta_j)_{j\in\mathbb{Z}}$ is supported on the set of indices $\{\eta,\dots,\alpha+\eta\}$ and therefore $\vartheta_w(\Delta_\bullet)$ is properly defined for $0\leqslant w<\alpha$. Then $v\leqslant v_p(\vartheta_\alpha(\Delta_\bullet))\leqslant v_p(\Delta_j)$ for all $j\in\mathbb{Z}$, and

$$\begin{aligned} &\textstyle\sum_i(\Delta_i - D_i)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha} \\ &\quad\equiv_{\mathfrak{I}} -\textstyle\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\alpha)/(p-1)} D_i\bullet_{\overline{\mathbb{Q}}_p} h_{i(p-1)+\alpha} \\ &\qquad -\textstyle\sum_{i\geqslant(t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)} D_i\bullet_{\overline{\mathbb{Q}}_p} h^*_{t-i(p-1)-\alpha} \\ &\qquad + E\bullet_{\overline{\mathbb{Q}}_p}\theta^{\alpha+1}h + F\bullet_{\overline{\mathbb{Q}}_p} h' + \mathsf{O}(p^\eta), \end{aligned} \tag{43}$$

for some polynomials h, h' and some $E, F\in\overline{\mathbb{Z}}_p$ with $v_p(E)\geqslant v$ and $v_p(F)>v$.

Proof. By using the equation

$$\gamma\bullet_{\overline{\mathbb{Q}}_p} v \equiv_{\mathfrak{I}} \mathfrak{a}^{-1}T(\gamma\bullet_{\overline{\mathbb{Q}}_p} v), \tag{44}$$

and equation (18), we can deduce that

$$\begin{aligned}
&\textstyle\sum_i(\Delta_i - D_i)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\
&\quad\equiv_{\mathfrak{I}} \mathfrak{a}^{-1}T\left(\textstyle\sum_i(\Delta_i - D_i)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\right)\\
&\quad\equiv_{\mathfrak{I}} \mathfrak{a}^{-1}\textstyle\sum_i(\Delta_i - D_i)\sum_{\lambda\in\mathbb{F}_p^\times}\left(\begin{smallmatrix} p & [\lambda]\\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}(-[\lambda]x+py)^{t-i(p-1)-\alpha}\\
&\qquad\quad + \mathfrak{a}^{-1}\textstyle\sum_i(\Delta_i - D_i)\left(p^{t-i(p-1)-\alpha}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right) + p^{i(p-1)+\alpha}\left(\begin{smallmatrix} 1 & 0\\ 0 & p\end{smallmatrix}\right)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\
&\quad\equiv_{\mathfrak{I}} \mathfrak{a}^{-1}\textstyle\sum_i(\Delta_i - D_i)\sum_{\lambda\in\mathbb{F}_p^\times}\left(\begin{smallmatrix} p & [\lambda]\\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}(-[\lambda]x+py)^{t-i(p-1)-\alpha}\\
&\qquad\quad - \textstyle\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\alpha)/(p-1)} D_i\bullet_{\overline{\mathbb{Q}}_p} h_{i(p-1)+\alpha}\\
&\qquad\quad - \textstyle\sum_{i\geqslant(t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)} D_i\bullet_{\overline{\mathbb{Q}}_p} h^*_{t-i(p-1)-\alpha} + \mathsf{O}(p^\eta). \qquad (45)
\end{aligned}$$

The third congruence follows from lemma 6. We also have

$$\begin{aligned}
&\textstyle\sum_{\lambda\in\mathbb{F}_p^\times}\left(\begin{smallmatrix} p & [\lambda]\\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}(-[\lambda]x+py)^{t-i(p-1)-\alpha}\\
&\quad\equiv_{\mathfrak{I}} \textstyle\sum_{\xi=0}^{2\eta}\binom{t-i(p-1)-\alpha}{\xi}p^\xi\sum_{\lambda\in\mathbb{F}_p^\times}[-\lambda]^{t-\alpha-\xi}\left(\begin{smallmatrix} p & [\lambda]\\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{t-\xi}y^\xi + \mathsf{O}\big(p^{2\eta}\big)\\
&\quad\equiv_{\mathfrak{I}} \mathfrak{a}\textstyle\sum_{\xi=0}^{2\eta}\binom{t-i(p-1)-\alpha}{\xi}\bullet_{\overline{\mathbb{Q}}_p}\sum_{\lambda\in\mathbb{F}_p^\times}[-\lambda]^{t-\alpha-\xi}\left(\begin{smallmatrix} 1 & [\lambda]\\ 0 & 1\end{smallmatrix}\right)h^*_\xi + \mathsf{O}\big(p^{2\eta}\big). \qquad (46)
\end{aligned}$$

The second congruence follows from lemma 6. By assumption, if

$$X_\xi = \textstyle\sum_i(\Delta_i - D_i)\binom{t-i(p-1)-\alpha}{\xi}, \qquad (47)$$

then $v_p(X_\xi) > v$ for $\xi\in\{0,\dots,\alpha\}$, and $v_p(X_\xi)\geqslant v$ for $\xi\in\{\alpha+1,\dots,2\eta\}$. This means that equation (46) implies that

$$\begin{aligned}
&\mathfrak{a}^{-1}\textstyle\sum_i(\Delta_i - D_i)\sum_{\lambda\in\mathbb{F}_p^\times}\left(\begin{smallmatrix} p & [\lambda]\\ 0 & 1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha}(-[\lambda]x+py)^{t-i(p-1)-\alpha}\\
&\quad\equiv_{\mathfrak{I}} \textstyle\sum_{\xi=0}^{2\eta} X_\xi\bullet_{\overline{\mathbb{Q}}_p}\sum_{\lambda\in\mathbb{F}_p^\times}[-\lambda]^{t-\alpha-\xi}\left(\begin{smallmatrix} 1 & [\lambda]\\ 0 & 1\end{smallmatrix}\right)h^*_\xi + \mathsf{O}(p^\eta), \qquad (48)
\end{aligned}$$

which together with equation (45) implies equation (43) with

$$\begin{aligned}
E\theta^{\alpha+1}h &= \textstyle\sum_{\xi=\alpha+1}^{2\eta} X_\xi\sum_{\lambda\in\mathbb{F}_p^\times}[-\lambda]^{t-\alpha-\xi}\left(\begin{smallmatrix} 1 & [\lambda]\\ 0 & 1\end{smallmatrix}\right)h^*_\xi,\\
Fh' &= \textstyle\sum_{\xi=0}^{\alpha} X_\xi\sum_{\lambda\in\mathbb{F}_p^\times}[-\lambda]^{t-\alpha-\xi}\left(\begin{smallmatrix} 1 & [\lambda]\\ 0 & 1\end{smallmatrix}\right)h^*_\xi. \qquad (49)
\end{aligned}$$

∎

Lemma 8. *Let $(C_l)_{l\in\mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$. Suppose that $\alpha\in\{0,\dots,\eta\}$ and $\beta\in\{\alpha,\dots,\eta\}$ and $v\in\mathbb{Q}$ and the family $(D_i)_{i\in\mathbb{Z}}$ defined by*

$$D_i = \left[i\in\left\{\left\lceil\tfrac{-\alpha}{p-1}\right\rceil,\dots,\left\lfloor\tfrac{t-\alpha}{p-1}\right\rfloor\right\}\right]D'_i + \textstyle\sum_{l=\alpha-\beta}^{\alpha} C_l\binom{t-\alpha+l}{i(p-1)+l} \qquad (50)$$

satisfy

$$\begin{aligned}
&v\leqslant v_p(\vartheta_w(D_\bullet)) \text{ for } \alpha\leqslant w\leqslant 2\delta,\\
&v < v_p(\vartheta_w(D_\bullet)) \text{ for } 0\leqslant w<\alpha. \qquad (51)
\end{aligned}$$

Note that $(D_i)_{i\in\mathbb{Z}}$ *is supported on the finite set of indices* $\{\lceil\frac{-\alpha}{p-1}\rceil,\dots,\lfloor\frac{t-\alpha}{p-1}\rfloor\}$. *Then*

$$\begin{aligned}&(1-p)^{-\alpha}\vartheta_\alpha(D_\bullet)\bullet_{\overline{\mathbb{Q}}_p}\theta^\alpha x^{\eta(p-1)}y^{t-\alpha(p+1)-\eta(p-1)}\\&\quad\equiv_{\mathfrak{I}}\tfrac{\mathfrak{a}p^{-\alpha}}{p-1}\textstyle\sum_{l=\alpha-\beta}^{\alpha}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\alpha-l}\\&\qquad+\textstyle\sum_i D'_i\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\qquad-\textstyle\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\alpha)/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h_{i(p-1)+\alpha}\\&\qquad-\textstyle\sum_{i\geqslant(t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h^*_{t-i(p-1)-\alpha}\\&\qquad+E\bullet_{\overline{\mathbb{Q}}_p}\theta^{\alpha+1}h+F\bullet_{\overline{\mathbb{Q}}_p}h'+\mathsf{O}(p^\eta),\end{aligned}\tag{52}$$

for some polynomials h,h' *and some* $E,F\in\overline{\mathbb{Z}}_p$ *with* $v_p(E)\geqslant v$ *and* $v_p(F)>v$.

Proof. Lemma 6 implies that

$$\begin{aligned}&\textstyle\sum_i(D_i-D'_i)\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\quad\equiv_{\mathfrak{I}}\tfrac{\mathfrak{a}p^{-\alpha}}{p-1}\textstyle\sum_{l=\alpha-\beta}^{\alpha}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\alpha-l}+\mathsf{O}(p^\eta).\end{aligned}\tag{53}$$

Equation (53) together with lemma 7 implies that

$$\begin{aligned}&\textstyle\sum_i\Delta_i\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\quad\equiv_{\mathfrak{I}}\tfrac{\mathfrak{a}p^{-\alpha}}{p-1}\textstyle\sum_{l=\alpha-\beta}^{\alpha}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\alpha-l}\\&\qquad+\textstyle\sum_i D'_i\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\qquad-\textstyle\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\alpha)/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h_{i(p-1)+\alpha}\\&\qquad-\textstyle\sum_{i\geqslant(t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h^*_{t-i(p-1)-\alpha}\\&\qquad+E\bullet_{\overline{\mathbb{Q}}_p}\theta^{\alpha+1}h+F\bullet_{\overline{\mathbb{Q}}_p}h'+\mathsf{O}(p^\eta),\end{aligned}\tag{54}$$

for some polynomials h,h' and some $E,F\in\overline{\mathbb{Z}}_p$ with $v_p(E)\geqslant v$ and $v_p(F)>v$. Equation (54) can be rewritten in the form of equation (52) because

$$\begin{aligned}&\textstyle\sum_i\Delta_i\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\quad=(1-p)^{-\alpha}\vartheta_\alpha(D_\bullet)\bullet_{\overline{\mathbb{Q}}_p}\theta^\alpha x^{\eta(p-1)}y^{t-\alpha(p+1)-\eta(p-1)}.\end{aligned}\tag{55}$$

∎

4. Proofs

We want to prove theorem M' by computing $\overline{\Theta}$. We accomplish this as the cumulative result of the following six subsections.

4.1. If Q is an ∞-dimensional factor of $\overline{\Theta}$, then Q is not a factor of $\widehat{N_\alpha}$, for $\alpha\in\{0,\dots,\varrho-1\}$. For $\alpha\in\{0,\dots,\varrho-1\}$, let us define the matrix

$$M_\alpha^{(r)}=\left(\binom{r-\alpha+j}{i(p-1)+j}\right)_{\{i\,|\,i(p-1)+\alpha\in(\varrho,r-\varrho)\},\,\alpha-\varrho\leqslant j\leqslant\alpha}.\tag{56}$$

So the rows of $M_\alpha^{(r)}$ are indexed by those i such that $i(p-1)+\alpha \in (\varrho, r-\varrho)$, and the columns of $M_\alpha^{(r)}$ are indexed by those j such that $\alpha-\varrho \leqslant j \leqslant \alpha$. This means that $M_\alpha^{(r)}$ has $C=\varrho+1$ columns and

$$R \leqslant \lfloor \tfrac{r-2\varrho+p-2}{p-1} \rfloor \leqslant \varrho+1 = C \tag{57}$$

rows, i.e. $M_\alpha^{(r)}$ has no more rows than columns. Let

$$M_\alpha^{(r)\prime} = \left(\tbinom{r-\alpha+j}{i(p-1)+j} \right)_{\{i \,|\, i(p-1)+\alpha \in (\varrho, r-\varrho)\},\, \alpha-R<j\leqslant\alpha} \tag{58}$$

be the right $R\times R$ submatrix of $M_\alpha^{(r)}$. We can write

$$\tbinom{r-\alpha+j}{i(p-1)+j} = \tbinom{r}{i(p-1)+\alpha} \tbinom{i(p-1)+\alpha}{\alpha-j} \tbinom{r}{\alpha-j}^{-1}, \tag{59}$$

so the $\mathbb{Z}_p$-module determined by the image of the matrix $M_\alpha^{(r)\prime}$ contains the $\mathbb{Z}_p$-module determined by the image of the matrix

$$\left(\tbinom{r}{i(p-1)+\alpha} \tbinom{i(p-1)+\alpha}{\alpha-j} \right)_{\{i \,|\, i(p-1)+\alpha \in (\varrho, r-\varrho)\},\, \alpha-R<j\leqslant\alpha}. \tag{60}$$

Lemma 9 implies that

$$v_p \left(\tbinom{r}{i(p-1)+\alpha} \right) \leqslant \lfloor \log_p(r+1) \rfloor = \mathcal{E}, \tag{61}$$

so the latter $\mathbb{Z}_p$-module contains $p^{\mathcal{E}} \times$ the $\mathbb{Z}_p$-module determined by the image of the matrix

$$M_\alpha^{(r)\prime\prime} = \left(\tbinom{i(p-1)+\alpha}{\alpha-j} \right)_{\{i \,|\, i(p-1)+\alpha \in (\varrho, r-\varrho)\},\, \alpha-R<j\leqslant\alpha}. \tag{62}$$

There exists a $\gamma \in \mathbb{Z}_{\geqslant 0}$ such that $M_\alpha^{(r)\prime\prime}$ is obtained from

$$M_\alpha^{(r)\prime\prime\prime} = \left(\tbinom{i(p-1)+\gamma}{j} \right)_{0\leqslant i,j<R} \tag{63}$$

by permuting the rows. By Vandermonde's convolution formula,

$$M_\alpha^{(r)\prime\prime\prime} = \left(\tbinom{i(p-1)}{j} \right)_{0\leqslant i,j<R} \cdot \left(\tbinom{\gamma}{j-i} \right)_{0\leqslant i,j<R}. \tag{64}$$

Since the matrix

$$\left(\tbinom{\gamma}{j-i} \right)_{0\leqslant i,j<R} \tag{65}$$

is upper triangular with 1's on the diagonal and

$$\det \left(\tbinom{i(p-1)}{j} \right)_{0\leqslant i,j<R} = (p-1)^R \det \left(\tbinom{i}{j} \right)_{0\leqslant i,j<R} = (p-1)^R \tag{66}$$

by a variant of Vandermonde's determinant identity, the reduction modulo p of $M_\alpha^{(r)\prime\prime\prime}$ has full rank (in characteristic p). This in turn implies that the reduction modulo p of $M_\alpha^{(r)\prime\prime}$ has full rank. Therefore, for each u such that

$$u(p-1)+\alpha \in (\varrho, r-\varrho), \tag{67}$$

there exist constants $C_\alpha(r,u), \ldots, C_{\alpha-R+1}(r,u)$ such that

$$M_\alpha^{(r)\prime} \left(C_\alpha(r,u), \ldots, C_{\alpha-R+1}(r,u) \right)^T = p^{\mathcal{E}} \left([i=u] \right)_{\{i \,|\, i(p-1)+\alpha \in (\varrho, r-\varrho)\}}, \tag{68}$$

i.e. such that

$$\textstyle\sum_{l=\alpha-R+1}^{\alpha} C_l(r,u) \tbinom{r-\alpha+l}{i(p-1)+l} = [i=u] p^{\mathcal{E}} \tag{69}$$

for all i such that

$$i(p-1)+\alpha \in (\varrho, r-\varrho). \tag{70}$$

By adding linear combinations of equation (69) for varying u, we get that

$$\begin{aligned}&\sum_{i(p-1)+\alpha\in(\varrho,r-\varrho)}\sum_{l=\alpha-\varrho}^{\alpha}C_l\tbinom{r-\alpha+l}{i(p-1)+l}x^{i(p-1)+\alpha}y^{r-i(p-1)-\alpha}\\&\quad+\sum_{i(p-1)+\alpha\in[0,\varrho]\cup[r-\varrho,r]}D_i'x^{i(p-1)+\alpha}y^{r-i(p-1)-\alpha}\\&\qquad=p^{\mathcal{E}}\theta^{\alpha}x^{p-1}y^{r-\alpha(p+1)-p+1}\in\widetilde{\Sigma}_r,\end{aligned}\tag{71}$$

for some C_l, D_i'. Let $D_i(r)$ be the coefficient of $x^{i(p-1)+\alpha}y^{r-i(p-1)-\alpha}$ on the right side of equation (71). Then, due to part (5) of lemma 6 and lemma 7 in [Ars21a],

$$\vartheta_w(D_\bullet(r))=\sum_i D_i(r)\tbinom{i(p-1)}{w}\tag{72}$$

is zero for $0\leqslant w<\alpha$, and has valuation that is greater than or equal to $\mathcal{E}$ for $w\geqslant\alpha$, with equality for $w=\alpha$. Let D_i be the coefficient of $x^{i(p-1)+\alpha}y^{r-i(p-1)-\alpha}$ in

$$\begin{aligned}&\sum_{i(p-1)+\alpha\in(\varrho,t-\varrho)}\sum_{l=\alpha-\varrho}^{\alpha}C_l\tbinom{t-\alpha+l}{i(p-1)+l}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\quad+\sum_{i(p-1)+\alpha\in[0,\varrho]\cup[t-\varrho,t]}D_i''x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha},\end{aligned}\tag{73}$$

where $D_i''=D_i'$ and $D_{t-i}''=D_{r-i}'$ for $i(p-1)+\alpha\in[0,\varrho]$. Since

$$v_p((i(p-1)+\alpha)!)\leqslant v_p(\varrho!)\leqslant k\tag{74}$$

for $i(p-1)+\alpha\in[0,\varrho]$, it is easy to show by using lemma 5 in [Ars21a] that

$$\vartheta_w(D_\bullet)=\vartheta_w(D_\bullet(r))+\mathsf{O}\big(\epsilon p^{-k-W}\big)\tag{75}$$

for all $0\leqslant w\leqslant W$. In particular, we can apply lemma 8 to the constants $(D_i)_{i\in\mathbb{Z}}$ and to $v=\mathcal{E}$, and as a result get that

$$\begin{aligned}&(1-p)^{-\alpha}\vartheta_\alpha(D_\bullet)\bullet_{\overline{\mathbb{Q}}_p}\theta^{\alpha}x^{\eta(p-1)}y^{t-\alpha(p+1)-\eta(p-1)}\\&\equiv_{\mathfrak{I}}\tfrac{\mathfrak{a}p^{-\alpha}}{p-1}\sum_{l=\alpha-\varrho}^{\alpha}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\alpha-l}\\&\qquad+\sum_{i(p-1)+\alpha\in[0,\varrho]\cup[t-\varrho,t]}D_i'''\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\alpha}y^{t-i(p-1)-\alpha}\\&\qquad-\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\alpha)/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h_{i(p-1)+\alpha}\\&\qquad-\sum_{i\geqslant(t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h^*_{t-i(p-1)-\alpha}\\&\qquad+E\bullet_{\overline{\mathbb{Q}}_p}\theta^{\alpha+1}h+F\bullet_{\overline{\mathbb{Q}}_p}h'+\mathsf{O}(p^{\eta}),\end{aligned}\tag{76}$$

for some h,h' and some $E,F\in\overline{\mathbb{Z}}_p$ with $v_p(E)\geqslant\mathcal{E}$ and $v_p(F)>\mathcal{E}$. Here

$$D_i'''=D_i''-\sum_{l=\alpha-\varrho}^{\alpha}C_l\tbinom{t-\alpha+l}{i(p-1)+l}\tag{77}$$

for all i such that

$$i(p-1)+\alpha\in[0,\varrho]\cup[t-\varrho,t].\tag{78}$$

The left side of equation (76) is $p^{\mathcal{E}}\psi$, where ψ is an integral element whose reduction modulo p represents a generator of $\widehat{N}_\alpha$. We can use lemma 6 to get that the first and second lines on the right side of equation (76) are

$$\mathsf{O}\big(p^{v_p(\mathfrak{a})-\varrho}\big)=\mathsf{O}\big(p^{\mathcal{E}}\big).\tag{79}$$

We can also use equation (71) to get that

$$D_i=\mathsf{O}\big(p^{\mathcal{E}}\big)\tag{80}$$

for all i such that

$$i(p-1)+\alpha \in [0, r-\varrho) \cup (t-r+\varrho, t]. \tag{81}$$

This is because

$$D_i = D_i(r) + \mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big) \tag{82}$$

for all i such that $i(p-1)+\alpha \in [0, r-\varrho)$, and

$$D_{t-i} = D_{r-i}(r) + \mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big) \tag{83}$$

for all i such that $i(p-1)+\alpha \in (t-r+\varrho, t]$. We also have, due to equation (71),

$$D_w = \mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big) \tag{84}$$

for all i and all

$$w \in \mathbb{Z}_{\leqslant 0} \cup \mathbb{Z}_{\geqslant \frac{t-2\alpha}{p-1}}. \tag{85}$$

This, together with lemma 6, implies that the sum of the third and fourth lines on the right side of equation (76) is $p^{\mathcal{E}} \times$ an integral element whose reduction modulo p represents the trivial element of $\widehat{N}_\alpha$. Finally, the fifth line of on the right side of equation (76) is evidently $p^{\mathcal{E}} \times$ an integral element whose reduction modulo p represents the trivial element of $\widehat{N}_\alpha$. Therefore equation (76) gives an element in $\mathscr{I}$ that generates $\widehat{N}_\alpha$, implying that no ∞-dimensional factor of $\overline{\Theta}$ is a subquotient of $\widehat{N}_\alpha$.

4.2. **If Q is an ∞-dimensional factor of $\overline{\Theta}$, then Q is not a factor of $\widehat{N_\alpha}$ for $\alpha \in \{\varrho+1, \ldots, \nu-1\}$.** Let

$$\varrho' = \lceil \tfrac{r-\alpha}{p} \rceil - 1. \tag{86}$$

For $\alpha \in \{\varrho+1, \ldots, \nu-1\}$ and $l \in \{\alpha-\varrho', \ldots, \alpha\}$ let us define

$$C_l = \Lambda_{\varrho'}(\alpha, l)\tbinom{r}{\alpha-l}. \tag{87}$$

As in subsection 4.1 we can conclude that

$$\textstyle\sum_{i(p-1)+\alpha \in (\alpha, \varrho'(p-1)+\alpha]} \sum_{l=\alpha-\varrho'}^{\alpha} C_l \binom{r-\alpha+l}{i(p-1)+l} x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha} = 0 \in \widetilde{\Sigma}_r. \tag{88}$$

Let D_i be the coefficient of $x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha}$ in

$$\textstyle\sum_{i(p-1)+\alpha \in (\alpha, t-r+\varrho'(p-1)+\alpha]} \sum_{l=\alpha-\varrho'}^{\alpha} C_l \binom{t-\alpha+l}{i(p-1)+l} x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha}. \tag{89}$$

Then it is easy to show by using lemma 5 in [Ars21a] that

$$\vartheta_w(D_\bullet) = \mathsf{O}\big(\epsilon p^{-k-W}\big) \tag{90}$$

for all $0 \leqslant w \leqslant W$. In particular, we can apply lemma 8 to the constants $(D_i)_{i \in \mathbb{Z}}$ and to $v = \eta$, and as a result get that

$$\begin{aligned} &\tfrac{\mathfrak{a}p^{-\alpha}}{p-1} \textstyle\sum_{l=\alpha-\varrho'}^{\alpha} C_l p^l \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-l} \left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h_{\alpha-l} \\ &\quad \equiv_{\mathfrak{I}} \textstyle\sum_{i(p-1)+\alpha \in [0,\alpha] \cup (t-r+\varrho'(p-1)+\alpha, t]} D_i' \bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha} \\ &\qquad + \textstyle\sum_{i \leqslant (\eta+v_p(\mathfrak{a})-\alpha)/(p-1)} D_i \bullet_{\overline{\mathbb{Q}}_p} h_{i(p-1)+\alpha} \\ &\qquad + \textstyle\sum_{i \geqslant (t-\alpha-\eta-v_p(\mathfrak{a}))/(p-1)} D_i \bullet_{\overline{\mathbb{Q}}_p} h^*_{t-i(p-1)-\alpha} + \mathsf{O}(p^\eta), \end{aligned} \tag{91}$$

where

$$D_i' = \sum_{l=\alpha-\varrho'}^{\alpha} C_l \tbinom{t-\alpha+l}{i(p-1)+l} \tag{92}$$

for all i such that

$$i(p-1)+\alpha \in [0,\alpha] \cup (t-r+\varrho'(p-1)+\alpha, t]. \tag{93}$$

By approximating D_i with $D_i(r) + \mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big)$ as in subsection 4.1 we can show that the third and fourth lines of equation (91) are in

$$\begin{aligned} \mathsf{O}\big(p^{\alpha p - v_p(\mathfrak{a})}\big) &= \mathsf{O}\big(\mathfrak{a}p^{-\alpha+(k-2v_p(\mathfrak{a})-p+3)}\big) \\ &= \mathsf{O}\big(\mathfrak{a}p^{-\alpha+(p-3)(k/(p-1)-1)}\big) \\ &= \mathsf{O}\big(\mathfrak{a}p^{-\alpha+2\mathcal{E}}\big). \end{aligned} \tag{94}$$

Consequently we get that

$$\begin{aligned} &\textstyle\sum_{i(p-1)+\alpha\in[0,\alpha]\cup(t-r+\varrho'(p-1)+\alpha,t]} \sum_{l=\alpha-\varrho'}^{\alpha} C_l \tbinom{t-\alpha+l}{i(p-1)+l} \bullet_{\overline{\mathbb{Q}}_p} x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha} \\ &\qquad \equiv_{\mathfrak{I}} \tfrac{\mathfrak{a}p^{-\alpha}}{p-1} \textstyle\sum_{l=\alpha-\varrho'}^{\alpha} C_l p^l \sum_{\mu\in\mathbb{F}_p^\times} [\mu]^{-l} \left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h_{\alpha-l} + \mathsf{O}\big(\mathfrak{a}p^{-\alpha+2\mathcal{E}}\big). \end{aligned} \tag{95}$$

Lemma 6 and the definition of $(C_l)_{\alpha-\varrho'\leqslant l\leqslant\alpha}$ then imply that

$$\begin{aligned} &\textstyle\sum_{i(p-1)+\alpha\in[0,\alpha]} X_i \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h_{i(p-1)+\alpha} \\ &\quad + \textstyle\sum_{i(p-1)+\alpha\in(t-r+\varrho'(p-1)+\alpha,t]} X_i^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h^*_{t-i(p-1)-\alpha} \\ &\qquad\quad \equiv_{\mathfrak{I}} \tfrac{1}{p-1} \textstyle\sum_{l=\alpha-\varrho'}^{\alpha} C_l p^l \sum_{\mu\in\mathbb{F}_p^\times} [\mu]^{-l} \left(\begin{smallmatrix} p & [\mu] \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{Q}}_p} h_{\alpha-l} + \mathsf{O}\big(p^{2\mathcal{E}}\big), \end{aligned} \tag{96}$$

where

$$\begin{aligned} X_i &= p^{-i(p-1)} \tbinom{t}{i(p-1)+\alpha} \tbinom{\varrho'-i}{\varrho'}, \\ X_i^* &= p^{i(p-1)+2\alpha-t} \tbinom{t}{i(p-1)+\alpha} \tbinom{\varrho'-i}{\varrho'}. \end{aligned} \tag{97}$$

By lemma 10,

$$v_p(X_0) = v_p\left(\tbinom{t}{\alpha}\right) < v_p\left(p^{-i(p-1)} \tbinom{t}{i(p-1)+\alpha} \tbinom{\varrho'-i}{\varrho'}\right) = v_p(X_i) \tag{98}$$

for all i such that $i(p-1)+\alpha \in [0,\alpha)$. By lemma 11,

$$v_p(X_0) = v_p\left(\tbinom{t}{\alpha}\right) < v_p\left(p^{i(p-1)+2\alpha-t} \tbinom{t}{i(p-1)+\alpha} \tbinom{\varrho'-i}{\varrho'}\right) = v_p(X_i^*) \tag{99}$$

for all i such that $i(p-1)+\alpha \in (t-r+\varrho'(p-1)+\alpha, t]$. By lemma 12,

$$v_p(X_0) = v_p\left(\tbinom{t}{\alpha}\right) < v_p\left(\Lambda_{\varrho'}(\alpha,l) \tbinom{t}{\alpha-l} p^l\right) = v_p\left(C_l p^l\right) \tag{100}$$

for all $l \in \{\alpha-\varrho', \ldots, \alpha\}$. Moreover, by lemma 9,

$$v_p(X_0) = v_p\left(\tbinom{t}{\alpha}\right) = v_p\left(\tbinom{r}{\alpha}\right) < 2\mathcal{E}. \tag{101}$$

So if we divide both sides of equation (96) by $\binom{t}{\alpha}$ we get an integral element, and if we reduce that integral element modulo p then the only contributing term to the result is the "$i=0$" term in the first line of equation (96). Therefore we can conclude that $\mathscr{I}$ contains

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \bullet_{\overline{\mathbb{F}}_p} h_\alpha, \tag{102}$$

which represents a generator of $\widehat{N}_\alpha$, and we can conclude the desired result.

4.3. **If $r-\varrho(p+1)=p-2$ and Q is an ∞-dimensional factor of $\overline{\Theta}$, then Q is not a factor of $\operatorname{ind}^G\operatorname{sub}(\varrho)$.** Let $r-\varrho(p+1)=p-2$, so that

$$\operatorname{sub}(\varrho)\cong\sigma_{p-2}(\varrho). \tag{103}$$

The proof in this case is very similar to the proof in subsection 4.1, so we just give a rough sketch. We let

$$M^{(r)}=\Big(\tbinom{r-\varrho+j}{i(p-1)+j}\Big)_{\{i\,|\,i(p-1)+\varrho\in(\varrho,r-\varrho)\},\,0\leqslant j\leqslant\varrho}. \tag{104}$$

As in subsection 4.1 we can prove that the image of a certain lattice under the right square submatrix of $M^{(r)}$ (seen as an endomorphism) contains $p^{\mathcal{E}}\times$ that lattice. We can conclude the following analogous equation to equation (71):

$$\begin{aligned}\textstyle\sum_{i(p-1)+\varrho\in(\varrho,r-\varrho)}\sum_{l=0}^{\varrho}C_l\tbinom{r-\varrho+l}{i(p-1)+l}x^{i(p-1)+\varrho}y^{r-i(p-1)-\varrho}&\\ +\textstyle\sum_{i(p-1)+\varrho\in[0,\varrho]\cup[r-\varrho]}D'_ix^{i(p-1)+\varrho}y^{r-i(p-1)-\varrho}&\\ =p^{\mathcal{E}}\theta^{\varrho}y^{r-\varrho(p+1)}\in\widetilde{\Sigma}_r,&\end{aligned} \tag{105}$$

for some integers D'_i. The main difference with equation (71) is that we must write $\theta^{\varrho}y^{r-\varrho(p+1)}$ instead of $\theta^{\varrho}x^{p-1}y^{r-\varrho(p+1)-p+1}$. This means that we can only conclude that

$$D_w=\mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big) \tag{106}$$

for all

$$w\in\mathbb{Z}_{<0}\cup\mathbb{Z}_{\geqslant\frac{t-2\varrho}{p-1}} \tag{107}$$

(rather than for all $w\in\mathbb{Z}_{\leqslant0}\cup\mathbb{Z}_{\geqslant\frac{t-2\varrho}{p-1}}$) in the equation

$$\begin{aligned}&(1-p)^{-\varrho}\vartheta_{\varrho}(D_\bullet)\bullet_{\overline{\mathbb{Q}}_p}\theta^{\varrho}x^{\eta(p-1)}y^{t-\varrho(p+1)-\eta(p-1)}\\ &\quad\equiv_{\mathfrak{I}}\tfrac{\mathfrak{a}p^{-\varrho}}{p-1}\textstyle\sum_{l=0}^{\varrho}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\varrho-l}\\ &\qquad+\textstyle\sum_{i(p-1)+\varrho\in[0,\varrho]\cup[t-\varrho,t]}D'''_i\bullet_{\overline{\mathbb{Q}}_p}x^{i(p-1)+\varrho}y^{t-i(p-1)-\varrho}\\ &\qquad-\textstyle\sum_{i\leqslant(\eta+v_p(\mathfrak{a})-\varrho)/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h_{i(p-1)+\varrho}\\ &\qquad-\textstyle\sum_{i\geqslant(t-\varrho-\eta-v_p(\mathfrak{a}))/(p-1)}D_i\bullet_{\overline{\mathbb{Q}}_p}h^*_{t-i(p-1)-\varrho}\\ &\qquad+E\bullet_{\overline{\mathbb{Q}}_p}\theta^{\varrho+1}h+F\bullet_{\overline{\mathbb{Q}}_p}h'+\mathsf{O}(p^{\eta}),\end{aligned} \tag{108}$$

for some h,h' and some $E,F\in\overline{\mathbb{Z}}_p$ with $v_p(E)\geqslant\mathcal{E}$ and $v_p(F)>\mathcal{E}$, which is the analogous equation to equation (75). In other words, the difference is that D_0 is not negligible, and instead

$$D_0=\vartheta_{\varrho}(D_\bullet)+\mathsf{O}\big(\epsilon p^{-v_p(\eta!)}\big). \tag{109}$$

So upon dividing equation (108) by $\vartheta_{\varrho}(D_\bullet)$ and reducing modulo p we get that $\mathscr{I}$ contains

$$1\bullet_{\overline{\mathbb{F}}_p}(\theta^{\varrho}x^{\eta(p-1)}y^{t-\varrho(p+1)-\eta(p-1)}+h_{\varrho}). \tag{110}$$

It is easy to show that his represents a generator of $\operatorname{ind}^G\operatorname{sub}(\varrho)$ (but is trivial in $\operatorname{ind}^G\operatorname{quot}(\varrho)$), which finishes the proof of the desired result as in subsection 4.1.

4.4. **If $r-\varrho(p+1)=1$ and Q is an ∞-dimensional factor of $\mathrm{ind}^G\,\mathrm{quot}(\varrho)$, then Q is a factor of $\mathrm{ind}^G\,\mathrm{quot}(\varrho)/T_{\mathrm{q},\varrho}$.** Let $r-\varrho(p+1)=1$, so that

$$\mathrm{quot}(\varrho)\cong\sigma_{p-2}(\varrho+1). \tag{111}$$

We want to show that $\mathscr{I}$ contains a representative of a generator of

$$T_{\mathrm{q},\varrho}\left(\mathrm{ind}^G\,\mathrm{quot}(\varrho)\right). \tag{112}$$

Let

$$C_l=\Lambda_\varrho(\varrho,l)\tbinom{r}{\varrho-l} \tag{113}$$

for $l\in\{0,\dots,\varrho\}$. As in subsection 4.2 we can conclude that

$$\begin{aligned}&\textstyle\sum_{i(p-1)+\varrho\in[0,\varrho]}X_i\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{i(p-1)+\varrho}\\&\quad+\textstyle\sum_{i(p-1)+\varrho\in(t-r+\varrho p,t]}X_i^*\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h^*_{t-i(p-1)-\varrho}\\&\qquad\equiv_{\mathfrak{I}}\tfrac{1}{p-1}\textstyle\sum_{l=0}^{\varrho}C_lp^l\sum_{\mu\in\mathbb{F}_p^\times}[\mu]^{-l}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{Q}}_p}h_{\varrho-l}+\mathsf{O}\big(p^{2\mathcal{E}}\big),\end{aligned} \tag{114}$$

where

$$\begin{aligned}X_i&=p^{-i(p-1)}\tbinom{t}{i(p-1)+\varrho}\tbinom{\varrho-i}{\varrho},\\X_i^*&=p^{(i-\varrho)(p-1)-1}\tbinom{t}{i(p-1)+\varrho}\tbinom{\varrho-i}{\varrho}.\end{aligned} \tag{115}$$

Again, by lemma 13,

$$v_p(X_0)=v_p\left(\tbinom{t}{\varrho}\right)<v_p\left(p^{-i(p-1)}\tbinom{t}{i(p-1)+\varrho}\tbinom{\varrho-i}{\varrho}\right)=v_p(X_i) \tag{116}$$

for all i such that $i(p-1)+\varrho\in[0,\varrho)$. By lemma 14,

$$v_p(X_0)=v_p\left(\tbinom{t}{\varrho}\right)<v_p\left(p^{(i-\varrho)(p-1)-1}\tbinom{t}{i(p-1)+\varrho}\tbinom{\varrho-i}{\varrho}\right)=v_p(X_i^*) \tag{117}$$

for all i such that $i(p-1)+\varrho\in(t-r+\varrho p,t]$. By lemma 15,

$$v_p(X_0)=v_p\left(\tbinom{t}{\varrho}\right)<v_p\left(\Lambda_\varrho(\varrho,l)\tbinom{t}{\varrho-l}p^l\right)=v_p\left(C_lp^l\right) \tag{118}$$

for all $l\in\{1,\dots,\varrho\}$. And, by lemma 9,

$$v_p(X_0)=v_p\left(\tbinom{t}{\varrho}\right)=v_p\left(\tbinom{r}{\varrho}\right)<2\mathcal{E}. \tag{119}$$

This means that if we divide both sides of equation (114) by $\binom{t}{\varrho}$ and reduce the resulting integral element modulo p, the two contributing terms are the "$i=0$" term in the first line of equation (114) and the "$l=0$" term in the third line of equation (114). Therefore $\mathscr{I}$ contains

$$\textstyle\sum_{\mu\in\mathbb{F}_p}\left(\begin{smallmatrix}p&[\mu]\\0&1\end{smallmatrix}\right)\bullet_{\overline{\mathbb{F}}_p}h_\varrho, \tag{120}$$

which is a representative of a generator of

$$T_{\mathrm{q},\varrho}\left(\mathrm{ind}^G\,\mathrm{quot}(\varrho)\right), \tag{121}$$

and that completes the proof.

Proof of theorem M' ($\Leftrightarrow$ theorem M). Let Q be an infinite-dimensional factor of $\overline{\Theta}_{t+2,\mathfrak{a}}$. Subsections 4.1 and 4.2 imply the following two facts about Q.

1. Q is not a factor of $\widehat{N}_\alpha$ for $\alpha\in\{0,\dots,\varrho-1\}$.

2. Q is not a factor of $\widehat{N}_\alpha$ for $\alpha\in\{\varrho+1,\dots,\nu-1\}$.

From these two facts we can conclude that either Q is a factor of $\operatorname{ind}^G \operatorname{sub}(\varrho)$ or it is a factor of $\operatorname{ind}^G \operatorname{quot}(\varrho)$. In light of theorem 5 and as in section 10 of [Ars21a] this implies equation (28) for $r - \varrho(p+1) \in \{-1, \dots, p-1\} \backslash \{-1, 1, p-2\}$.

Subsections 4.3 and 4.4 prove the following facts for $r - \varrho(p+1) \in \{1, p-2\}$.

3. If

$$r - \varrho(p+1) = p - 2 \tag{122}$$

(and therefore $\operatorname{sub}(\varrho) \cong \sigma_{p-2}(\varrho)$) then Q is a factor of $\operatorname{ind}^G \operatorname{quot}(\varrho)$.

4. If

$$r - \varrho(p+1) = 1 \tag{123}$$

(and therefore $\operatorname{quot}(\varrho) \cong \sigma_{p-2}(\varrho + 1)$) and Q is a factor of $\operatorname{ind}^G \operatorname{quot}(\varrho)$ then Q is a factor of of $\operatorname{ind}^G \operatorname{quot}(\varrho)/T_{\mathrm{q},\varrho}$.

These two claims imply equation (28) for $r - \varrho(p+1) \in \{1, p-2\}$. Since we assume that $p + 1 \nmid k - 1$, we have $r - \varrho(p+1) \neq -1$, and therefore the proof is complete. ■

Proof that theorem M implies corollary C. As in subsection 4.2 of [BLZ04] we can use theorem M to conclude that μ_l is supported on

$$\left[0, \tfrac{1}{p+1} + \tfrac{\log_p l}{l-1}\right] \cup \left[\tfrac{p}{p+1} - \tfrac{\log_p l}{l-1}, 1\right], \tag{124}$$

and that completes the proof because

$$\tfrac{\log_p l}{l-1} \to 0 \tag{125}$$

as $l \to \infty$—we omit the details. ■

5. Combinatorics

Lemma 9. *If $\alpha \in \mathbb{Z}_{\geqslant 1}$ and $\beta \in \{0, \dots, \alpha\}$ then*

$$v_p\left(\tbinom{\alpha}{\beta}\right) \leqslant \lfloor \log_p \alpha \rfloor. \tag{126}$$

Proof. A theorem by Kummer says that

$$v_p\left(\tbinom{\alpha}{\beta}\right) \tag{127}$$

is the number of times one carries over a digit when adding β and $\alpha - \beta$, and is therefore strictly less than the number $\lfloor \log_p \alpha \rfloor + 1$ of digits of α. ■

Let $\alpha \in \{\varrho, \dots, \nu - 1\}$, let

$$\varrho' = \lceil \tfrac{r-\alpha}{p} \rceil - 1, \tag{128}$$

and for $l \in \{\alpha - \varrho', \dots, \alpha\}$ let

$$C_l = \Lambda_{\varrho'}(\alpha, l)\tbinom{r}{\alpha - l}. \tag{129}$$

Note that $\varrho' \leqslant \varrho$ and if $r = \varrho'(p+1) + 1$ and $\alpha = \varrho$ then $\varrho' = \varrho$. The constants C_l are precisely those constants that satisfy

$$\textstyle\sum_{l=\alpha-\varrho'}^{\alpha} C_l \tbinom{r}{\alpha-l}^{-1} \tbinom{(p-1)X+\alpha}{\alpha-l} = \tbinom{\varrho' - X}{\varrho'} \in \mathbb{Q}_p[X]. \tag{130}$$

Moreover, let

$$X_i = p^{-i(p-1)}\tbinom{r}{i(p-1)+\alpha}\tbinom{\varrho'-i}{\varrho'} \text{ for } i \in \mathbb{Z},$$
$$X_i^* = p^{i(p-1)+2\alpha-r}\tbinom{r}{i(p-1)+\alpha}\tbinom{\varrho'-i}{\varrho'} \text{ for } i \in \mathbb{Z}. \tag{131}$$

Lemma 10. *If $\alpha > \varrho$ and $i(p-1)+\alpha \in [0,\alpha)$ then*

$$v_p\left(X_0\right) < v_p\left(X_i\right). \tag{132}$$

Proof. Note that $i < 0$, so let us write $j = -i > 0$. We have

$$X_0 = \tbinom{r}{\alpha},$$
$$X_i = p^{j(p-1)}\tbinom{r}{\alpha-j(p-1)}\tbinom{\varrho'+j}{\varrho'} = p^{j(p-1)}\tbinom{r}{\alpha}\frac{\alpha_{j(p-1)}}{(r-\alpha+j(p-1))_{j(p-1)}}\tbinom{\varrho'+j}{j}. \tag{133}$$

Therefore we want to show that

$$v_p\left((r-\alpha+j(p-1))_{j(p-1)}\right) < v_p\left(\alpha_{j(p-1)}\right) + v_p\left((\varrho'+j)_j\right) + j(p-1) - v_p(j!). \tag{134}$$

Note that $v_p(j!) \leqslant \frac{j}{p-1}$, so it is enough to show that

$$v_p\left((r-\alpha+j(p-1))_{j(p-1)}\right) < v_p\left(\alpha_{j(p-1)}\right) + v_p\left((\varrho'+j)_j\right) + j\left(p-1-\tfrac{1}{p-1}\right). \tag{135}$$

We have

$$\lfloor \tfrac{r-\alpha+j(p-1)}{p} \rfloor \leqslant \varrho' + j = \lceil \tfrac{r-\alpha+jp}{p} \rceil - 1. \tag{136}$$

We also have

$$\lceil \tfrac{r-\alpha+1}{p} \rceil \geqslant \varrho' + 1 = \lceil \tfrac{r-\alpha}{p} \rceil. \tag{137}$$

Therefore each term of the product

$$(r-\alpha+j(p-1))_{j(p-1)} = (r-\alpha+j(p-1))\cdots(r-\alpha+1) \tag{138}$$

that is divisible by p is p times a term of the product

$$(\varrho'+j)_j = (\varrho'+j)\cdots(\varrho'+1), \tag{139}$$

implying that

$$v_p\left((r-\alpha+j(p-1))_{j(p-1)}\right) \leqslant v_p\left((\varrho'+j)_j\right) + w, \tag{140}$$

where w is the number of terms of the product in equation (138) that are divisible by p. Equation (135) follows from the fact that

$$w \leqslant \lfloor \tfrac{j(p-1)+p-1}{p} \rfloor < j\left(p-1-\tfrac{1}{p-1}\right). \tag{141}$$

■

Lemma 11. *If $\alpha > \varrho$ and $i(p-1)+\alpha \in (\varrho'(p-1)+\alpha, r]$ then*

$$v_p\left(X_0\right) < v_p\left(X_i^*\right). \tag{142}$$

Proof. We have $i(p-1)+\alpha \in (\varrho'(p-1)+\alpha, r]$ and therefore

$$j = i(p-1)+2\alpha-r \in \left[\lceil \tfrac{r-\alpha}{p} \rceil(p-1)+2\alpha-r, \alpha\right] \subseteq \left[\tfrac{(p+1)\alpha-r}{p}, \alpha\right] \subseteq (0,\alpha]. \tag{143}$$

So we can write

$$X_0 = \binom{r}{\alpha},$$
$$X_i^* = p^{i(p-1)+2\alpha-r}\binom{r}{i(p-1)+\alpha}\binom{\varrho'-i}{\varrho'} = p^j\binom{r}{\alpha}\frac{\alpha_j}{(r-\alpha+j)_j}\binom{\varrho'-i}{\varrho'}. \tag{144}$$

Therefore we want to show that

$$v_p\left((r-\alpha+j)_j\right) < v_p\left(\alpha_j\right) + v_p\left(\binom{\varrho'-i}{\varrho'}\right) + j. \tag{145}$$

We have

$$\binom{\varrho'-i}{\varrho'} = (-1)^{\varrho'}\binom{i-1}{\varrho'} = (-1)^{\varrho'}\binom{i-1}{i-\varrho'-1} = (-1)^{\varrho'}\frac{(i-1)\cdots(\varrho'+1)}{(i-\varrho'-1)!}. \tag{146}$$

Because $\alpha > \varrho'$ and $i > \varrho'$, we have $r \leqslant ip+\alpha-1$, and therefore $\alpha - j \leqslant i-1$. Moreover, $\alpha \geqslant \varrho+1 \geqslant \varrho'+1$. Therefore the union of the intervals $(\varrho'+1, i-1]$ and $(\alpha-j+1,\alpha]$ is a single interval. This implies that

$$v_p\left(\alpha_j(i-1)_{i-\varrho'-1}\right) \geqslant v_p\left(\max\{i-1,\alpha\}\cdots\min\{\varrho'+1,\alpha-j\}\right). \tag{147}$$

We also have

$$v_p\left((i-\varrho'-1)!\right) \leqslant \tfrac{i-\varrho'-1}{p-1}. \tag{148}$$

Since

$$p\max\{i-1,\alpha\}+p-1 \geqslant r-\alpha+j \text{ and } p\min\{\varrho'+1,\alpha-j\}-p+1 \leqslant r-\alpha+1, \tag{149}$$

we have

$$v_p\left((r-\alpha+j)_j\right) \leqslant v_p\left(\max\{i-1,\alpha\}\cdots\min\{\varrho'+1,\alpha-j\}\right) + \tfrac{j+p-1}{p}. \tag{150}$$

So it is enough to show that

$$j > \tfrac{j+p-1}{p} + \tfrac{i-\varrho'-1}{p-1}, \tag{151}$$

which follows from

$$\tfrac{(i(p-1)+2\alpha-r)(p-1)}{p} \geqslant \tfrac{i-\varrho'+p-2}{p-1}. \tag{152}$$

The latter inequality follows from the fact that $\gamma(i) = \frac{(i(p-1)+2\alpha-r)(p-1)}{p} - \frac{i-\varrho'+p-2}{p-1}$ is increasing in i and $\gamma(\varrho'+1) \geqslant \frac{2(p-1)}{p} - 1 > 0$. ∎

Lemma 12. *If $\alpha > \varrho$ and $l \in \{\alpha-\varrho', \ldots, \alpha\}$ then*

$$v_p\left(X_0\right) < v_p\left(C_l p^l\right). \tag{153}$$

Proof. For $l \in \{\alpha-\varrho', \ldots, \alpha\}$ let

$$C_l' = C_l\binom{r}{\alpha-l}^{-1} \in \mathbb{Z}_p, \tag{154}$$

so that

$$\textstyle\sum_{j=\alpha-\varrho'}^{\alpha} C_j'\binom{(p-1)X+\alpha}{\alpha-j} = \binom{\varrho'-X}{\varrho'} \in \mathbb{Q}_p[X]. \tag{155}$$

We have

$$X_0 = \binom{r}{\alpha},$$
$$C_l p^l = C_l'\binom{r}{\alpha-l}p^l = C_l'\binom{r}{\alpha}\frac{\alpha_l}{(r-\alpha+l)_l}p^l. \tag{156}$$

So we want to show that

$$v_p\left(C_l'\right) > v_p\left((r-\alpha+l)_l\right) - v_p\left(\alpha_l\right) - l. \tag{157}$$

First suppose that $\alpha(p+1)+p-1 \geqslant r+l$. Then $p\alpha+p-1 \geqslant r-\alpha+l$, implying that the largest term in the sequence $(r-\alpha+l,\dots,r-\alpha+1)$ that is divisible by p is at most as large as p times the largest term in the sequence $(\alpha,\dots,\alpha-l+1)$. Moreover,

$$p\alpha-pl+1 \leqslant p\alpha-p(\alpha-\varrho')+1 = p\varrho'+1 \leqslant r-\alpha+1, \tag{158}$$

implying that the smallest term in the sequence $(r-\alpha+l,\dots,r-\alpha+1)$ that is divisible by p is at least as large as p times the smallest term in the sequence $(\alpha,\dots,\alpha-l+1)$. Therefore

$$v_p\left((r-\alpha+l)_l\right)-v_p\left(\alpha_l\right)-l \leqslant w-l \leqslant \lfloor \tfrac{l+p-1}{p} \rfloor - l \leqslant 0, \tag{159}$$

where w is the number of terms in the sequence $(r-\alpha+l,\dots,r-\alpha+1)$ that are divisible by p. Equality in equation (159) holds if and only if $l=1$ and $r-\alpha+1=p\alpha$, which can never happen. So the right side of equation (157) is negative, implying equation (153) in the case when $\alpha(p+1)+p-1 \geqslant r+l$. Now suppose that $\alpha(p+1)+p \leqslant r+l$, so that

$$l \geqslant p(\alpha-\varrho')+p-1 \geqslant 2p-1. \tag{160}$$

For $j \in \{\alpha-\varrho',\dots,\alpha\}$ let

$$C''_j = (-1)^{\varrho'}(p-1)^{j-\alpha}C'_j\varrho'_{j-\alpha+\varrho'} \in \mathbb{Z}_p, \tag{161}$$

so that

$$\textstyle\sum_{j=\alpha-\varrho'}^{\alpha} C''_j \prod_{u=0}^{\alpha-j-1}\left(X+\frac{\alpha-u}{p-1}\right) = (X-1)\cdots(X-\varrho') \in \mathbb{Q}_p[X]. \tag{162}$$

We use the fact that among any l consecutive integers there can be at most one whose valuation is at least $\log_p l$. Moreover, if there is such a term then the sum of the valuations of all the other terms is at most $\frac{l-1}{p-1}$, and if there is no such term then the sum of the valuations of all the terms is at most $\frac{l-1}{p-1}+\lfloor \log_p l\rfloor$. So if there is no term in the sequence $(r-\alpha+l,\dots,r-\alpha+1)$ whose valuation is at least $\log_p l$, then

$$v_p\left((r-\alpha+l)_l\right)-l < \lfloor \log_p l \rfloor - \tfrac{l(p-2)}{p-1} \leqslant 0, \tag{163}$$

so

$$v_p\left(C'_l\tbinom{r}{\alpha}\tfrac{\alpha_l}{(r-\alpha+l)_l}p^l\right) \geqslant v_p\left(\tbinom{r}{\alpha}\tfrac{1}{(r-\alpha+l)_l}p^l\right) > v_p\left(\tbinom{r}{\alpha}\right). \tag{164}$$

Suppose now that there is a term in the sequence $(r-\alpha+l,\dots,r-\alpha+1)$ whose valuation is $\gamma \geqslant \log_p l$. If $\gamma \leqslant \frac{l(p-2)}{p-1}$ then we can similarly deduce that

$$v_p\left(C'_l\tbinom{r}{\alpha}\tfrac{\alpha_l}{(r-\alpha+l)_l}p^l\right) \geqslant v_p\left(\tbinom{r}{\alpha}\tfrac{1}{(r-\alpha+l)_l}p^l\right) > v_p\left(\tbinom{r}{\alpha}\right), \tag{165}$$

so suppose that $\gamma \geqslant \frac{l(p-2)}{p-1} \geqslant \log_p l$. If the term in $(r-\alpha+l,\dots,r-\alpha+1)$ whose valuation is γ is p times a term in $(\alpha,\dots,\alpha-l+1)$, then we can similarly deduce that

$$v_p\left(C'_l\tbinom{r}{\alpha}\tfrac{\alpha_l}{(r-\alpha+l)_l}p^l\right) \geqslant v_p\left(\tbinom{r}{\alpha}\tfrac{\alpha_l}{(r-\alpha+l)_l}p^l\right) > v_p\left(\tbinom{r}{\alpha}\right), \tag{166}$$

so suppose that the term in $(r-\alpha+l,\dots,r-\alpha+1)$ whose valuation is γ is in the subsequence $(r-\alpha+l,\dots,p(\alpha+1))$. Let $q=p^{\gamma-1}$. Informally speaking, the assumptions that q is a "large" power of p and that $(p(\varrho'+1)+l,\dots,p(\alpha+1))$

contains a multiple of pq (say bpq) and that $\alpha > \varrho'$ imply that α and ϱ' are "just below" a multiple of q, and

$$\varrho' + l/p + 1 \geqslant bq > \alpha > \varrho' \text{ and } q \geqslant p^{\frac{l(p-2)}{p-1}-1} \text{ and } l \geqslant 2p-1. \tag{167}$$

For $u \in \{0, \ldots, \varrho'-1\}$ let z_u be the integer in $\{1, \ldots, q\}$ that is congruent to $\frac{u-\alpha}{p-1}$ modulo q, and for $j \in \{\alpha-\varrho', \ldots, \alpha\}$ let C_j''' be such that

$$\sum_{j=\alpha-\varrho'}^{\alpha} C_j''' \prod_{u=0}^{\alpha-j-1} (X - z_u) = \prod_{i=1}^{q-s}(X-i)^b \prod_{i=q-s+1}^{q}(X-i)^{b-1} \in \mathbb{Q}_p[X], \tag{168}$$

where $s = bq - \varrho'$. By reducing equation (162) modulo q we get $C_j'' \equiv C_j''' \bmod q$. For $j \in \{\alpha-\varrho', \ldots, \alpha\}$, let $F_j(X) = \prod_{u=0}^{\alpha-j-1}(X - z_u)$. Then

$$F_\alpha(X) \mid \cdots \mid F_{\alpha-\varrho'}(X), \tag{169}$$

and if $i_0 \in \{\alpha-\varrho', \ldots, \alpha\}$ is the smallest index such that $F_{i_0}(X)$ divides

$$G(X) = \prod_{i=1}^{q-s}(X-i)^b \prod_{i=q-s+1}^{q}(X-i)^{b-1}, \tag{170}$$

then equation (168) implies that $C_j''' = 0$ for all $j \in \{i_0+1, \ldots, \alpha\}$, and therefore $C_j'' \equiv 0 \bmod q$ for all $j \in \{i_0+1, \ldots, \alpha\}$. We want to show that $i_0 > l$, i.e. that

$$\prod_{u=0}^{\alpha-l}(X - z_u) \mid G(X). \tag{171}$$

We have

$$\prod_{u=0}^{(b-1)q-1}(X - z_u) = \prod_{i=1}^{q}(X-i)^{b-1}, \tag{172}$$

so in order to show equation (171) it is enough to show that

$$\prod_{u=(b-1)q}^{\alpha-l}(X - z_u) \mid \prod_{i=1}^{q-s}(X-i), \tag{173}$$

i.e. that the sets

$$\{bq - \alpha + i \,|\, i \in \{0, \ldots, \alpha - l - (b-1)q\}\} \tag{174}$$

and

$$\{q - (p-1)i \,|\, i \in \{0, \ldots, bq - \varrho' - 1\}\} \tag{175}$$

are disjoint. This follows from

$$\begin{aligned} l &\geqslant p(bq - \varrho' - 1) \\ &\implies l > (p-1)(bq - \varrho' - 1) \\ &\implies q - l < q - (p-1)(bq - \varrho' - 1). \end{aligned} \tag{176}$$

So $i_0 > l$, and therefore $C_l'' \equiv 0 \bmod q$. Since

$$\begin{aligned} X_0 &= \tbinom{r}{\alpha}, \\ C_l p^l &= C_l' \tbinom{r}{\alpha} \tfrac{\alpha_l}{(r-\alpha+l)_l} p^l = C_l'' \tbinom{r}{\alpha} \tfrac{\alpha\cdots(\varrho'+1)}{(r-\alpha+l)_l} p^l, \end{aligned} \tag{177}$$

and since $v_p(C_l'') \geqslant \gamma - 1$ and

$$v_p\left((r-\alpha+l)_l\right) \leqslant \gamma + \tfrac{l-1}{p-1} \tag{178}$$

and

$$l - \tfrac{l-1}{p-1} - 1 = \tfrac{(l-1)(p-2)}{p-1} \geqslant 2(p-2) > 0, \tag{179}$$

we can deduce equation (153) and complete the proof. ∎

Lemma 13. *If* $r = \varrho(p+1)+1$ *and* $\alpha = \varrho$ *and* $i(p-1)+\varrho \in [0,\varrho)$ *then*

$$v_p\left(X_0\right) < v_p\left(X_i\right). \tag{180}$$

Proof. The proof is similar to the proof of lemma 10. Note that $i < 0$, so let us write $j = -i > 0$. We have

$$\begin{aligned} X_0 &= \tbinom{r}{\varrho}, \\ X_i &= p^{j(p-1)}\tbinom{r}{\varrho-j(p-1)}\tbinom{\varrho+j}{\varrho} = p^{j(p-1)}\tbinom{r}{\varrho}\tfrac{\varrho_{j(p-1)}}{(\varrho p+j(p-1)+1)_{j(p-1)}}\tbinom{\varrho+j}{j}. \end{aligned} \tag{181}$$

Therefore we want to show that

$$v_p\left((\varrho p + j(p-1)+1)_{j(p-1)}\right) < v_p\left((\varrho+j)_{jp}\right) + j\left(p-1-\tfrac{1}{p-1}\right), \tag{182}$$

since $v_p(j!) \leqslant \frac{j}{p-1}$. Since

$$(\varrho+j)p > \varrho p + j(p-1)+1 \text{ and } \varrho p + 2 > (\varrho - j(p-1)+1)p, \tag{183}$$

each term of the product

$$(\varrho p + j(p-1)+1)_{j(p-1)} \tag{184}$$

that is divisible by p is p times a term of the product

$$(\varrho+j)_{jp}. \tag{185}$$

This together with

$$j\left(p-1-\tfrac{1}{p-1}\right) > \tfrac{(j+1)(p-1)}{p} \tag{186}$$

implies equation (182). ∎

Lemma 14. *If* $r = \varrho(p+1)+1$ *and* $\alpha = \varrho$ *and* $i(p-1)+\varrho \in (\varrho p, r]$ *then*

$$v_p\left(X_0\right) < v_p\left(X_i^*\right). \tag{187}$$

Proof. The proof is similar to the proof of lemma 11. We have $i(p-1)+\varrho \in (\varrho p, r]$ and therefore

$$j = (i-\varrho)(p-1)-1 \in [p-2, \varrho]. \tag{188}$$

So we can write

$$\begin{aligned} X_0 &= \tbinom{r}{\varrho}, \\ X_i^* &= p^j\tbinom{r}{\varrho}\tfrac{\varrho_j}{(\varrho p+j+1)_j}\tbinom{\varrho-i}{\varrho} = (-1)^{\varrho}p^j\tbinom{r}{\varrho}\tfrac{(i-1)_{i-\varrho+j-1}}{(\varrho p+j+1)_j(i-\varrho-1)!}, \end{aligned} \tag{189}$$

where the last equality follows as in the proof of lemma 11. Therefore we want to show that

$$v_p\left((\varrho p+j+1)_j\right) + \tfrac{i-\varrho-1}{p-1} < v_p\left((i-1)_{i-\varrho+j-1}\right) + j. \tag{190}$$

Since $ip-1 \geqslant i(p-1)+\varrho$ and $\varrho p - jp + 1 \leqslant \varrho p + 2$, we have

$$v_p\left((\varrho p+j+1)_j\right) \leqslant v_p\left((i-1)_{i-\varrho+j-1}\right) + \tfrac{j+p-1}{p}, \tag{191}$$

and equation (190) follows from

$$j > \tfrac{j+p-1}{p} + \tfrac{i-\varrho-1}{p-1}, \tag{192}$$

as in the proof of lemma 11. ∎

Lemma 15. *If $r = \varrho(p+1)+1$ and $\alpha = \varrho$ and $l \in \{1,\ldots,\varrho\}$ then*

$$v_p(X_0) < v_p\left(C_l p^l\right). \tag{193}$$

Proof. For $l \in \{0,\ldots,\varrho\}$ let

$$C'_l = C_l \binom{r}{\varrho-l}^{-1} \in \mathbb{Z}_p, \tag{194}$$

so that

$$\textstyle\sum_{j=0}^{\varrho} C'_j \binom{(p-1)X+\varrho}{\varrho-j} = \binom{\varrho-X}{\varrho} \in \mathbb{Q}_p[X]. \tag{195}$$

We have

$$\begin{aligned} X_0 &= \tbinom{r}{\varrho}, \\ C_l p^l &= C'_l \tbinom{r}{\varrho-l} p^l = C'_l \tbinom{r}{\varrho} \tfrac{\varrho_l}{(\varrho p+l+1)_l} p^l. \end{aligned} \tag{196}$$

Therefore in order to prove the lemma it is enough to show that, for $l \in \{1,\ldots,\varrho\}$,

$$v_p(C'_l) > v_p((\varrho p+l+1)_l) - v_p(\varrho_l) - l. \tag{197}$$

Note that equation (197) is not true for $l=0$, since both sides are zero. As in the proof of lemma 12, if $l \leqslant p-2$ then we can deduce that

$$v_p((\varrho p+l+1)_l) - v_p(\varrho_l) - l \leqslant \tfrac{l+p-1}{p} - l \leqslant 0, \tag{198}$$

with equality if and only if $l=1$ and $\varrho p+2 = \varrho p$, which can never happen. So let us assume that $l \geqslant p-1$. For $j \in \{0,\ldots,\varrho\}$ let

$$C''_j = (-1)^{\varrho}(p-1)^{j-\varrho} C'_j \varrho_j \in \mathbb{Z}_p, \tag{199}$$

so that

$$\textstyle\sum_{j=0}^{\varrho} C''_j \prod_{u=0}^{\varrho-j-1}\left(X + \frac{\varrho-u}{p-1}\right) = (X-1)\cdots(X-\varrho) \in \mathbb{Q}_p[X]. \tag{200}$$

As in the proof of lemma 12, if there is no term in $(\varrho p+l+1,\ldots,\varrho p+2)$ whose valuation is at least $\log_p l$, then

$$v_p((\varrho p+l+1)_l) - l < \lfloor \log_p l \rfloor - \tfrac{l(p-2)}{p-1} \leqslant 0, \tag{201}$$

implying equation (197). Suppose now that there is a term in

$$(\varrho p+l+1,\ldots,\varrho p+2) \tag{202}$$

whose valuation is $\gamma \geqslant \log_p l$. If $\gamma \leqslant \frac{l(p-2)}{p-1}$ then we can similarly deduce equation (197) as in the proof of lemma 12, so suppose that $\gamma \geqslant \frac{l(p-2)}{p-1} \geqslant \log_p l$. Let $q = p^{\gamma-1}$. We have, for some positive integer b,

$$\varrho + (l+1)/p \geqslant bq > \varrho \text{ and } q \geqslant p^{\frac{l(p-2)}{p-1}-1} \text{ and } l \geqslant p-1. \tag{203}$$

For $u \in \{0,\ldots,\varrho-1\}$ let z_u be the integer in $\{1,\ldots,q\}$ that is congruent to $\frac{u-\varrho}{p-1}$ modulo q, and for $j \in \{0,\ldots,\varrho\}$ let C'''_j be such that

$$\textstyle\sum_{j=0}^{\varrho} C'''_j \prod_{u=0}^{\varrho-j-1}(X-z_u) = \prod_{i=1}^{q-s}(X-i)^b \prod_{i=q-s+1}^{q}(X-i)^{b-1} \in \mathbb{Q}_p[X], \tag{204}$$

where $s = bq-\varrho$. By reducing equation (200) modulo q we get $C''_j \equiv C'''_j \bmod q$. For $j \in \{0,\ldots,\varrho\}$, let $F_j(X) = \prod_{u=0}^{\varrho-j-1}(X-z_u)$. Then

$$F_{\varrho}(X) \mid \cdots \mid F_0(X), \tag{205}$$

and if $i_0 \in \{0,\ldots,\varrho\}$ is the smallest index such that $F_{i_0}(X)$ divides

$$G(X) = \textstyle\prod_{i=1}^{q-s}(X-i)^b \prod_{i=q-s+1}^{q}(X-i)^{b-1}, \tag{206}$$

then equation (204) implies that $C_j''' = 0$ for all $j \in \{i_0 + 1, \ldots, \varrho\}$, and therefore $C_j'' \equiv 0 \bmod q$ for all $j \in \{i_0 + 1, \ldots, \varrho\}$. So again, as in the proof of lemma 12, we can complete the proof by noting that

$$\begin{aligned} l \geqslant p(bq - \varrho) - 1 & \\ \implies l > (p-1)(bq - \varrho - 1) & \\ \implies q - l < q - (p-1)(bq - \varrho - 1) & \\ \implies i_0 > l, & \end{aligned} \tag{207}$$

and therefore that, due to equation (196), equation (193) follows from

$$\begin{aligned} & v_p\left(C_l''\right) + v_p\left(\varrho_l\right) - v_p\left((\varrho p + l + 1)_l\right) - l \\ & \geqslant \gamma - 1 - \gamma - \tfrac{l-1}{p-1} + l = \tfrac{(l-1)(p-2)}{p-1} \geqslant \tfrac{(p-2)^2}{(p-1)} > 0. \end{aligned} \tag{208}$$

∎

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